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ON TWO MATRIX SYSTEMS DERIVED FROM A POLYNOMIAL OF EVEN ORDER WITH REAL COEFFICIENTS

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ABSTRACT

A polynomial f(z) of even order n can be written either in summation form with real coefficients A_o, A_1, ..., A_n, or in product form where the factors are quadratic polynomials in z. The coefficients of the factor polynomials are y_ν associated with z^1, and x_ν associated with z^o. A comparison of the coefficients in both forms yields systems of equations that can be systematically ordered. When x_ν is replaced by x, and y_ν by y, a Z matrix can be recognized as the essential part of one of the systems. Based on fundamental theorems of algebra, the Z matrix has been developed for polynomials of orders 2, 4, and 6.

The Z matrix of order n has a strong internal construction. Its relationship to matrices of the adjacent orders $n \pm 2$ is such that it can be obtained in ascending or descending sequence. The elements of the Z matrix are simultaneous polynomials in x and y. The Y matrix can be derived from a comparison of the coefficients or from the Z matrix. The elements of the Y matrix are polynomials in x only. The elements of the main diagonal of the Y matrix and the elements in the upper parallel to the diagonal can be obtained from the coefficients of f(x). A process of iterative differentiation yields the elements below the main diagonal; a process of iterative integration yields the elements above the upper parallel to the main diagonal. The Z matrix and Y matrix are thus known and can be computed for a polynomial of any even order with real coefficients. Properties and characteristics of the Z matrix and the Y matrix have been compiled in 12 theorems.

PREFACE

In the course of elaborating methods for finding the roots of polynomials with real coefficients, a comparison of the coefficients has disclosed some striking properties in certain systems of equations. It seems worthwhile to summarize these properties in a separate report. A careful search of the literature has not disclosed any previous publication of similar nature, which should justify presentation of this data as 'theorems.' The author will be happy to receive comments on the adequacy of his literature sources and citations.

ACKNOWLEDGMENTS

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ON TWO MATRIX SYSTEMS DERIVED FROM A POLYNOMIAL OF EVEN ORDER WITH REAL COEFFICIENTS

1. INTRODUCTION

The function

$$f(z) = \sum_{k=0}^{n} A_k z^k = A_n z^n + A_{n-1} z^{n-1} + A_{n-2} z^{n-2} + \ldots$$

$$\ldots + A_2 z^2 + A_1 z + A_0 \qquad (1.1\underline{a})$$

is a polynomial in z. Its order is n and, for our discussion, the coefficients A_k represent real numbers, either positive or negative. The subscript $\underline{k}$ is identical with the exponent $\underline{k}$ of the power z^k that is associated with the coefficient. The coefficient A_n is assumed to be $\equiv 1$. All polynomials will be understood as though written in the expanded form of Eq. (1.1a).

For reasons that will be explained immediately, we will deal only with polynomials of even order n. Thus, throughout the discussion n is an even integer. The polynomial equation

$$f(z) = 0 \qquad (1.2)$$

has n simple roots. Let us postulate intermediately that all the $n/2$ roots in Eq. (1.2) are conjugate complex pairs. Then

$$f(z) = \prod_{\nu=1}^{n/2} (z^2 + y_\nu z + x_\nu) \qquad (1.1\underline{b})$$

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is an alternative form of Eq. (1.1a). Each factor $z^2 + y_\nu z + x_\nu$ in Eq. (1.1b) represents one of the n/2 pairs of conjugate complex zeros

$$Z_\nu = \alpha_\nu \pm j\beta_\nu, \tag{1.3}$$

where

$$\alpha_\nu = -\tfrac{1}{2} y_\nu, \tag{1.3a}$$

$$\beta_\nu = \tfrac{1}{2}\sqrt{4x_\nu - y_\nu^2}, \tag{1.3b}$$

and

$$\nu = 1, 2, \ldots n/2. \tag{1.3c}$$

The coefficients y_ν and x_ν in Eq. (1.1b) represent real numbers. All x_ν are positive. The y_ν may be positive or negative, depending on whether the corresponding pair of conjugate complex zeros is in the left or right half of the complex z plane. If f(z) has only conjugate complex roots, then the coefficient A_0 in Eq. (1.1a) is positive.

If f(z) has only conjugate complex roots, it may be considered as the residual polynomial of a polynomial F(z) of order N (N being an even or odd integer) from which all the N-n real roots have been removed. By Sturm's[1] method we can find out if a polynomial has real roots and discover their approximate location. A very rough indication suffices as a start toward obtaining more accurate values by working out some real z values. Horner's[2] method is then used to find the real roots, within any desired accuracy. Removing these roots leaves a polynomial of order n that has only conjugate complex roots.

Schematic procedures for both Sturm's and Horner's methods are described by Willers,[3] as well as in a report now in preparation by the present author. The problem of root-finding will not be further discussed in this paper. What has been given here should be enough to show justification for dealing with polynomials of even order only. There is no necessity to postulate that the even-order polynomials under discussion have conjugate complex roots only.

2. COEFFICIENT COMPARISONS

From Eqs. (1.1a) and (1.1b),

$$\sum_{k=0}^{n} A_k z^k = \prod_{\nu=1}^{n/2} (z^2 + y_\nu z + x_\nu), \tag{2.1}$$

with $A_n \equiv 1$. After the multiplication on the right side of this equation is carried out, we compare the coefficients that are associated with the same power z^k on both sides of the equation. The result is the following

$$\left.\begin{aligned}
A_n &= \mathrm{comb}^0\, y_\nu \equiv 1 \\
A_{n-1} &= \mathrm{comb}^1\, y_\nu \\
A_{n-2} &= \mathrm{comb}^2\, y_\nu + \mathrm{comb}^1\, x_\nu \cdot \mathrm{comb}^0\, y_\nu \\
A_{n-3} &= \mathrm{comb}^3\, y_\nu + \mathrm{comb}^1\, x_\nu \cdot \mathrm{comb}^1\, y_\nu \\
A_{n-4} &= \mathrm{comb}^4\, y_\nu + \mathrm{comb}^1\, x_\nu \cdot \mathrm{comb}^2\, y_\nu + \mathrm{comb}^2\, x_\nu \cdot \mathrm{comb}^0\, y_\nu \\
A_{n-5} &= \mathrm{comb}^5\, y_\nu + \mathrm{comb}^1\, x_\nu \cdot \mathrm{comb}^3\, y_\nu + \mathrm{comb}^2\, x_\nu \cdot \mathrm{comb}^1\, y_\nu \\
&\vdots
\end{aligned}\right\} \tag{2.2}$$

In these formulas $\mathrm{comb}^2\, y_\nu$, for instance, represents the sum of all second-order products y_1y_2, y_1y_3, y_2y_3, etc., taken from the set of $(n/2)y_\nu$, omitting repetitive indices such as y_1y_1 or y_2y_2. Similarly, $\mathrm{comb}^1\, x_\nu \cdot \mathrm{comb}^1\, y_\nu$ represents the sum of all products x_1y_2, x_1y_3, x_2y_1, x_2y_3, etc., taken from the set of $(n/2)x_\nu$ and $(n/2)y_\nu$, excluding such combinations as x_1y_1 or x_2y_2. A combination comb^0 is defined as $\equiv 1$.

Equations (2.2) show the systematic development of the sequence of coefficients A_n, A_{n-1}, ... There is no disturbance in this systematic structure until we have passed the coefficient $A_{n/2}$. For the coefficient $A_{(n-2)/2}$, the first element in the sum corresponding to those in Eqs. 2.2 would be $\mathrm{comb}^{(n+2)/2}\, y_\nu$. For this combination, however, the only elements available are $(n/2)y_\nu$. This kind of combination is therefore $\equiv 0$. For the coefficient $A_0 = A_{n-n}$, especially, only one combination is possible, and that is $\mathrm{comb}^{n/2} x_\nu = \mathrm{comb}^0 y_\nu \cdot \mathrm{comb}^{n/2} x_\nu$.

Let us explain the development for the sixth-order polynomial as an example. Here,

$$z^6 + A_5 z^5 + A_4 z^4 + A_3 z^3 + A_2 z^2 + A_1 z + A_0$$
$$= (z^2+y_1 z+x_1)(z^2+y_2 z+x_2)(z^2+y_3 z+x_3).$$

From Eqs. (2.2),

$$A_n = A_6 \equiv 1$$
$$A_{n-1} = A_5 = \text{comb}^1 y_\nu = y_1 + y_2 + y_3$$
$$A_{n-2} = A_4 = \text{comb}^2 y_\nu + \text{comb}^1 x_\nu = y_1 y_2 + y_1 y_3 + y_2 y_3 + x_1 + x_2 + x_3$$
$$A_{n-3} = \text{comb}^3 y_\nu + \text{comb}^1 x_\nu \cdot \text{comb}^1 y_\nu$$
$$= y_1 y_2 y_3 + x_1(y_2+y_3) + x_2(y_1+y_3) + x_3(y_1+y_2).$$

The coefficient

$$A_{n-3} = A_3 = A_{6/2},$$

and since from here on only the three elements y_1, y_2, and y_3 are available, we have to watch how we apply Eqs. (2.2). In

$$A_{n-4} = A_2 = \text{comb}^4 y_\nu + \text{comb}^1 x_\nu \cdot \text{comb}^2 y_\nu + \text{comb}^2 x_\nu,$$

the element $\text{comb}^4 y_\nu \equiv 0$, and thus

$$A_2 = x_1 y_2 y_3 + x_2 y_1 y_3 + x_3 y_1 y_2 + x_1 x_2 + x_1 x_3 + x_2 x_3.$$

In

$$A_{n-5} = A_1 = \text{comb}^5 y_\nu + \text{comb}^1 x_\nu \cdot \text{comb}^3 y_\nu + \text{comb}^2 x_\nu \cdot \text{comb}^1 y_\nu,$$

$\text{comb}^5 y_\nu \equiv 0$ since five elements are not available. But the second element in the sum also disappears because $\text{comb}^1 x_\nu \cdot \text{comb}^3 y_\nu$ is not possible without repeating an index. (A combination $x_1 y_1 y_2 y_3$ is not allowed.) Thus, only the third term is valid and so

$$A_1 = x_1 x_2 y_3 + x_1 x_3 y_2 + x_2 x_3 y_1.$$

Finally, since $\text{comb}^6 y_\nu \equiv 0$ and $\text{comb}^4 y_\nu \equiv 0$, and $\text{comb}^2 x_\nu \cdot \text{comb}^2 y_\nu$ is not possible without repeating an index, all that remains is

$$A_0 = x_1 x_2 x_3.$$

The coefficients of the polynomials of orders 2, 4, and 6 are presented in Table 1 for convenience in following the discussion.

3. EQUATIONS DERIVED FROM COEFFICIENT COMPARISONS

3.1 A System of Equations Derived From Comparing Coefficients of the Fourth-Order Polynomial

We will now perform some very simple algebraic operations on the polynomial of order 4 with the coefficients listed in Table 1. We arbitrarily eliminate

$$x_2 = A_0/x_1 \tag{3.1}$$

and

$$y_2 = A_3 - y_1. \tag{3.2}$$

By substitution we obtain

$$A_2 = x_1 + A_0/x_1 + y_1(A_3 - y_1) \tag{3.3}$$

and

$$A_1 = x_1(A_3 - y_1) + A_0 y_1/x_1. \tag{3.4}$$

Note that the coefficient A_1 does not appear in Eq. (3.3) and that the coefficient A_2 does not appear in Eq. (3.4). By other selections for eliminating x_2 and y_2 and subsequent substitution we can find an equation in which the coefficient A_o does not appear, as well as an equation in which all four coefficients A_o through A_3 do appear. Each of these equations is a simultaneous polynomial in x_1 and y_1.

If we eliminate x_1 and y_1 instead of x_2 and y_2, the results will be the same except that the simultaneous polynomials obtained will be in x_2 and y_2. Since the subscripts are after all only indices of the factors in the sequential product of Eq. (1.1b), which are commutative, we may in all cases interchange x_1 and y_1 with x_2 and y_2. We therefore drop the indices 1 and 2 and let

x represent x_1 or x_2,

and

y represent y_1 or y_2,

noting that when x stands for x_1, then y stands for y_1; and if x stands for x_2, then y stands for y_2. Equations (3.3) and (3.4) and those mentioned in the paragraph following those equations now read as follows:

$$\begin{array}{llllll} & -x^3A_3 & +x^2yA_2 & -x(y^2-x)A_1 & +y(y^2-2x)A_0 & \equiv 0. \quad (3.5\underline{a}) \\ -x^3 & & +x^2A_2 & -xyA_1 & +(y^2-x)A_0 & \equiv 0. \quad (3.5\underline{b}) \\ -x^2y & +x^2A_3 & & -xA_1 & +yA_0 & \equiv 0. \quad (3.5\underline{c}) \\ -x(y^2-x) & +xyA_3 & -xA_2 & & +A_0 & \equiv 0. \quad (3.5\underline{d}) \\ -y(y^2-2x) & +(y^2-x)A_3 & -yA_2 & +A_1 & & \equiv 0. \quad (3.5\underline{e}) \end{array}$$

Each of the Eqs. (3.5a) to (3.5e) inclusive is a simultaneous polynomial in x and y, involving of course the real solutions $x = x_1$ and $x = x_2$, and $y = y_1$ and $y = y_2$, but written without the distinguishing index.

A closer look at these equations shows that the coefficients A_3 through A_0 have the leading role in this equation system. This becomes more evident when we represent the simultaneous polynomials by Z, using a double integer index ik, with i indicating the row and k indicating the column in the expressions given in Eqs. (3.5a) to (3.5e). We then obtain

$$Z_{00} + Z_{01}A_3 + Z_{02}A_2 + Z_{03}A_1 + Z_{04}A_0 \equiv 0. \qquad (3.6\underline{a})$$

$$Z_{10} + Z_{11}A_3 + Z_{12}A_2 + Z_{13}A_1 + Z_{14}A_0 \equiv 0. \qquad (3.6\underline{b})$$

$$Z_{20} + Z_{21}A_3 + Z_{22}A_2 + Z_{23}A_1 + Z_{24}A_0 \equiv 0. \qquad (3.6\underline{c})$$

$$Z_{30} + Z_{31}A_3 + Z_{32}A_2 + Z_{33}A_1 + Z_{34}A_0 \equiv 0. \qquad (3.6\underline{d})$$

$$Z_{40} + Z_{41}A_3 + Z_{42}A_2 + Z_{43}A_1 + Z_{44}A_0 \equiv 0. \qquad (3.6\underline{e})$$

These equations can be abbreviated by using matrix notation of the form

$$[Z_{ik}A_{4-k}]_{4,4} = [Z_{ik}]_{4,4} \cdot [A_{4-k}]_4 \equiv 0. \qquad (3.6)$$

The matrix in Eq. (3.6) will be referred to as the Z matrix. It is a square matrix of 5×5 elements (5 rows and 5 columns), numbered from Row 0 and Column 0, respectively. The elements of the Z matrix are simultaneous polynomials in x and y. In its full form the matrix is:

$$\begin{bmatrix} Z_{00} & Z_{01} & Z_{02} & Z_{03} & Z_{04} \\ Z_{10} & Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{20} & Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{30} & Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{40} & Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix} = \left[Z_{ik} \right]_{4,4}. \qquad (3.7)$$

If we now compare Eqs. (3.6a) through (3.6e) with Eqs. (3.5a) through (3.5e) to identify the elements Z_{ik} with the simultaneous polynomials, we discover the following significant properties of the matrix $\left[Z_{ik}\right]_{4,4}$:

1) The elements in the main diagonal are all $\equiv 0$.

$$Z_{00} = Z_{11} = Z_{22} = Z_{33} = Z_{44} \equiv 0. \qquad (3.8)$$

2) The elements $Z_{ik} = Z_{ki}$ if $(i+k)$ is odd. (3.9)

3) The elements $Z_{ik} = -Z_{ki}$ if $(i+k)$ is even. (3.10)

From Eqs. (3.9) and (3.10)

4)

$$Z_{ik} = -Z_{ki}(-1)^{i+k}. \qquad (3.11)$$

5) The structure in the diagonal direction is such that

$$\begin{aligned} Z_{01} &= -x^3 & Z_{02} &= x^2 y \\ Z_{12} &= -Z_{01}/x = x^2 & Z_{13} &= -Z_{02}/x = -xy \\ Z_{23} &= -Z_{12}/x = -x & Z_{24} &= -Z_{13}/x = y \\ Z_{34} &= -Z_{23}/x = +1 & & \end{aligned}$$

$$\begin{aligned} Z_{03} &= -x(y^2-x) & Z_{04} &= y(y^2-2x) \\ Z_{14} &= -Z_{03}/x = y^2-x & & \end{aligned}$$

Thus, generally,

$$Z_{(i+1)(k+1)} = - Z_{ik}/x. \tag{3.12}$$

Note that in any three sequential equations in the system Eqs. (3.5a) through (3.5e), the third results from operations on the two preceding it. For instance, if we multiply Eq. (3.5d) by -y and add the result to Eq. (3.5c), we obtain Eq. (3.5e) multiplied by -x.

6) The structure in the column direction is therefore such that

$$Z_{ik} = yZ_{(i+1)k} - xZ_{(i+2)k}. \tag{3.13}$$

It can easily be proved that if the elements in any column or in any row are known, Eqs. (3.8) through (3.13) will give all the elements of the Z matrix. Since, for example, $Z_{nn} = 0$ and $Z_{(n-1)n} = +1$, the last column is known from Eq. (3.13).

3.2 A System of Equations Derived From Comparing Coefficients of the Sixth-Order Polynomial

A procedure of elimination and subsequent substitution can be performed on the polynomial of order 6 with the coefficients listed in Table 1, just as in Sec. 3.1 for the polynomial of order 4. In this case,

x stands for x_1, x_2, or x_3,

y stands for y_1, y_2, or y_3.

When x represents x_1, then y represents y_1, and so on. The result is the following:

$$\left[Z_{ik}A_{6-k}\right]_{6,6} = \left[Z_{ik}\right]_{6,6} \cdot \left[A_{6-k}\right]_{6} \equiv 0. \tag{3.14}$$

Since the matrix $\left[Z_{ik}\right]_{6,6}$ represented by this equation has the same properties as described by statements (3.8) through (3.13), it is sufficient to present only the elements of row 0 of the Z matrix derived from the sixth-order polynomial:

$$\left.\begin{array}{l} Z_{01} = -x^5 \quad Z_{02} = x^4 y \quad Z_{03} = -x^3(y^2-x) \\ Z_{04} = x^2 y(y^2-2x) \quad Z_{05} = -x(y^4-3xy^2+x^2) \quad Z_{06} = y(y^4-4xy^2+3x^2) \end{array}\right\} \tag{3.15}$$

Note, for instance, that Z_{02} in the Z matrix of order 4 is different from Z_{02} in the Z matrix of order 6, just as A_2 in the polynomial of order 4 is different from A_2 in the polynomial of order 6. So long as we stay within the same order, however, further distinguishing indices may be omitted.

3.3 The Z Matrix of the Polynomial of Order 2

The equations that can be derived from the polynomial of order 2 are immediately given by the definition of the coefficients. They are—a triviality, of course—the following:

$$-xA_1 + yA_0 \equiv 0. \tag{3.16}$$

$$-x + A_0 \equiv 0. \tag{3.17}$$

$$-y + A_1 \equiv 0. \tag{3.18}$$

All three equations are satisfied by $x = x_1$ and $y = y_1$, since $A_0 = x_1$ and $A_1 = y_1$. The elements of the Z matrix are

$$\left.\begin{array}{c} Z_{00} = Z_{11} = Z_{22} \equiv 0 \\ Z_{01} = Z_{10} = -x; \quad Z_{02} = -Z_{20} = y \\ Z_{12} = Z_{21} = 1 \end{array}\right\}. \tag{3.19}$$

All the statements in Eqs. (3.8) through (3.13) are true for this trivial case also.

For convenient reference, the elements of the Z matrices of orders 2, 4, and 6 are listed in Table 2.

4. COMPARISON OF THE Z MATRICES OF THE POLYNOMIALS OF ORDERS 2, 4, AND 6

In Sec. 3 we showed the relationships between elements in the Z matrices of orders 2, 4, and 6. In each case the matrices had been derived from a comparison of the polynomial coefficients according to fundamental theorems of algebra. We will now compare the elements of one of these matrices with the elements of the adjacent orders. Let us first compare the matrix $\left[Z_{ik}\right]_{6,6}$ of order 6 with the matrix $\left[Z_{ik}\right]_{4,4}$ of order 4. Here we will use an additional subscript to indicate the matrix order containing the element being examined.

The elements of the matrix $\left[Z_{ik}\right]_{6,6}$ are listed in Table 2. If we delete Rows 5 and 6 and Columns 5 and 6 of the matrix $\left[Z_{ik}\right]_{6,6}$, the residual square matrix contains 5 rows and 5 columns. Dividing the elements of the residual matrix by x^2 gives the exact matrix $\left[Z_{ik}\right]_{4,4}$. For instance,

$$(Z_{01})_4 = (Z_{01})_6 / x^2 = -x^5/x^2 = -x^3,$$

$$(Z_{02})_4 = (Z_{02})_6 / x^2 = x^4 y/x^2 = x^2 y,$$

and so on.

We now consider the matrix $\left[Z_{ik}\right]_{4,4}$ of order 4. If we delete Rows 3 and 4 and Columns 3 and 4 of this matrix the residual square matrix contains 3 rows and 3 columns. Dividing each element of the residual matrix by x^2 gives the exact elements of the matrix $\left[Z_{ik}\right]_{2,2}$. Thus we find, at least for the order sequence 2, 4, and 6, that the elements of the matrix $\left[Z_{ik}\right]_{n,n}$ can be obtained from the elements of the matrix $\left[Z_{ik}\right]_{(n+2)(n+2)}$ by deleting the last two rows and the last two columns of the matrix of order (n+2) and dividing the elements of the residual square matrix by x^2.

Let us now reverse the procedure. Assume that the elements of the matrix $\left[Z_{ik}\right]_{4,4}$ are known and that the elements of the matrix $\left[Z_{ik}\right]_{6,6}$ are to be found. All the elements of the matrix $\left[Z_{ik}\right]_{6,6}$ in the area of the five Columns 0 to 4 and the five Rows 0 to 4 are known immediately, since we have only to multiply the corresponding elements of the matrix $\left[Z_{ik}\right]_{4,4}$ by x^2:

$$\begin{bmatrix} x^2(Z_{00})_4 & x^2(Z_{01})_4 & x^2(Z_{02})_4 & x^2(Z_{03})_4 & x^2(Z_{04})_4 & Z_{05} & Z_{06} \\ x^2(Z_{10})_4 & x^2(Z_{11})_4 & x^2(Z_{12})_4 & x^2(Z_{13})_4 & x^2(Z_{14})_4 & Z_{15} & Z_{16} \\ x^2(Z_{20})_4 & x^2(Z_{21})_4 & x^2(Z_{22})_4 & x^2(Z_{23})_4 & x^2(Z_{24})_4 & Z_{25} & Z_{26} \\ x^2(Z_{30})_4 & x^2(Z_{31})_4 & x^2(Z_{32})_4 & x^2(Z_{33})_4 & x^2(Z_{34})_4 & Z_{35} & Z_{36} \\ x^2(Z_{40})_4 & x^2(Z_{41})_4 & x^2(Z_{42})_4 & x^2(Z_{43})_4 & x^2(Z_{44})_4 & Z_{45} & Z_{46} \\ Z_{50} & Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} \\ Z_{60} & Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66} \end{bmatrix} = \left[Z_{ik}\right]_{6,6}.$$

To find the elements in Rows 5 and 6 and Columns 5 and 6, we apply the rules of the diagonal and of the column structure of the matrix $\left[Z_{ik}\right]_{6,6}$. From statement (3.12) we know that

$$(Z_{15})_6 = -(Z_{04})_6/x = -(Z_{04})_4 \cdot x = -xy(y^2-2x),$$

and that

$$(Z_{26})_6 = -(Z_{15})_6/x = y(y^2-2x).$$

Furthermore, since

$$(Z_{25})_6 = -(Z_{14})_6/x = -(Z_{14})_4 \cdot x = -x(y^2-x),$$

all the elements in Row 2 of the Matrix $\left[Z_{ik}\right]_{6,6}$ are now known. Proceeding in the same way, we find all the other elements in Rows 3, 4, 5, and 6 of Columns 5 and 6. For instance,

$$(Z_{36})_6 = (Z_{14})_6 / x^2 = (Z_{14})_4 = y^2 - x.$$

To find elements Z_{05}, Z_{06}, and Z_{16}, we apply statement (3.13):

$$\begin{aligned}
(Z_{05})_6 &= y(Z_{15})_6 - x(Z_{25})_6 \\
&= -xy^2(y^2-2x) + x^2(y^2-x) = -x(x^2-3xy^2+y^4), \\
(Z_{16})_6 &= y(Z_{26})_6 - x(Z_{36})_6 \\
&= y^2(y^2-2x) - x(y^2-x) = x^2-3y^2+y^4, \\
(Z_{06})_6 &= y(Z_{16})_6 - x(Z_{26})_6 \\
&= y(x^2-3xy^2+y^4) - xy(y^2-2x) = y(3x^2-4xy^2+y^4).
\end{aligned}$$

All the elements in Columns 5 and 6 are now known. We obtain the rest from statement (3.11) and thus complete the matrix $\left[Z_{ik}\right]_{6,6}$. A check shows that our results agree with the coefficients in Table 2.

As another example we now derive the matrix $\left[Z_{ik}\right]_{4,4}$ from the matrix $\left[Z_{ik}\right]_{2,2}$ without repeating the explanation.

$$\begin{bmatrix}
x^2(Z_{00})_2 & x^2(Z_{01})_2 & x^2(Z_{02})_2 & Z_{03} & Z_{04} \\
x^2(Z_{10})_2 & x^2(Z_{11})_2 & x^2(Z_{12})_2 & Z_{13} & Z_{14} \\
x^2(Z_{20})_2 & x^2(Z_{21})_2 & x^2(Z_{22})_2 & Z_{23} & Z_{24} \\
Z_{30} & Z_{31} & Z_{32} & Z_{33} & Z_{34} \\
Z_{40} & Z_{41} & Z_{42} & Z_{43} & Z_{44}
\end{bmatrix} = \left[Z_{ik}\right]_{4,4}.$$

$$(Z_{13})_4 = -(Z_{02})_4/x = -(Z_{02})_2\, x = -xy.$$

$$(Z_{24})_4 = -(Z_{13})_4/x = y$$

$$(Z_{23})_4 = -(Z_{12})_4/x = -(Z_{12})_2\, x = -x.$$

$$(Z_{34})_4 = -(Z_{23})_4/x = +1.$$

$$(Z_{03})_4 = y(Z_{13})_4 - x(Z_{23})_4 = -x(y^2-x).$$

$$(Z_{14})_4 = y(Z_{24})_4 - x(Z_{34})_4 = y^2 - x.$$

$$(Z_{04})_4 = y(Z_{14})_4 - x(Z_{24})_4 = y(y^2-2x).$$

The results agree with Table 2, which lists the results obtained *through fundamental theorems.*

We have thus shown, at least for the order sequence 2, 4, and 6, that the Z matrix of order n can be derived from the Z matrix of order n-2.

5. THE GENERAL PROGRESSIVE EVOLUTION OF THE Z MATRICES

The fundamental theorems for deriving the matrices $\left[Z_{ik}\right]_{4,4}$ and $\left[Z_{ik}\right]_{6,6}$ can also be used to derive Z matrices of higher order than 6. The higher the order, however, the more tedious the procedure becomes. It is much simpler to derive the Z matrices progressively, as in Sec. 4, where the matrix of order 6 was derived from that of order 4, and the matrix of order 4 from that of order 2. That the Z matrices through order 6 are consistently interrelated warrants the induction [4, 5] that we can derive the Z matrix of order 8 from that of order 6, the Z matrix of order 10 from that of order 8 and so on. This has not been taken for granted, however, and the results for all Z matrices through order 12 have been painstakingly proved step by step. No attempt has been made to find a general proof.

The step-by-step proof is relatively easy. The polynomial

$$\dot{f}(z) = \prod_{\nu=1}^{n/2} (z^2 + y_\nu z + x_\nu) \tag{5.1}$$

is first replaced by the polynomial

$$f_0(z) = (z^2 + yz + x)^{n/2}. \tag{5.2}$$

Equations (2.2) then change to the following:

$$\left.\begin{aligned}
A_n &\equiv 1 \\
A_{n-1} &= \binom{n/2}{1} y \\
A_{n-2} &= \binom{n/2}{2} y^2 + \binom{n/2}{1} x \\
A_{n-3} &= \binom{n/2}{3} y^3 + \binom{n/2}{1} x \binom{n/2-1}{1} y \\
A_{n-4} &= \binom{n/2}{4} y^4 + \binom{n/2}{1} x \binom{n/2-1}{2} y^2 + \binom{n/2}{2} x^2 \\
&\vdots
\end{aligned}\right\} \tag{5.3}$$

In the transformation of Eqs. (2.2) to Eqs. (5.3) the previous combinations of x_ν and y_ν are transformed to powers of x and y multiplied by binomial coefficients. By Eqs. (5.3) we can prove that if any row in the Z matrix of order n yields a consistent equation [as for instance Eqs. (3.5a) through (3.5e)] the row in the Z matrix of order n+2 will also yield a consistent equation, and vice versa. Both matrices can be developed progressively or retrogressively, as described in Sec. 4.

It is of course easier to use the retrogressive procedure, which yields the Z matrix of order n-2 from that of order n after the last two rows and columns are deleted and the residual elements divided by x^2. When a certain order is expected to be the limit for practical applications, it is sufficient to present only one row or one column of the limiting Z matrix. This will yield all the elements of the matrix, as well as all the elements of the Z matrices of lower order. Table 3 presents the elements of Row 0 of the Z matrix of order 12. All the elements of the Z matrices of orders 2 through 12 can be found from this table retrogressively. The Z matrix of any order higher than 12 can be found progressively.

The results are consistent. They have been proved up through order 12 and there is nothing to indicate that at the higher orders there can be any disturbance either in the Z matrix itself or in its interrelationship with its neighbors. In Sec. 6 we therefore venture to state some theorems about the structure of the Z matrix and its relationship between adjacent matrices.

6. THE Z MATRIX THEOREMS

Theorem 1

With $A_n \equiv 1$, given that

$$f(z) = \sum_{k=0}^{n} A_k z^k = \prod_{\nu=1}^{n/2} (z^2 + y_\nu z + x_\nu)$$

is a polynomial with real coefficients A_k and of even order. Comparisons of its coefficients in the sum-and-product form will yield a system of n+1 equations:

$$\left[Z_{ik}\right]_{n,n} \cdot \left[A_{n-k}\right]_n = 0.$$

The elements of the square matrix $\left[Z_{ik}\right]_{n,n}$ are simultaneous polynomials in x and y. The pair x, y stands for any pair x_ν, y_ν. The matrix is called the Z matrix.

Theorem 2

The elements Z_{ii} in the main diagonal of the Z matrix are zero. The other elements of the matrix are partly symmetric, partly skew symmetric, according to

$$Z_{ik} = - Z_{ki} (-1)^{i+k}.$$

Theorem 3

In the diagonal direction of the Z matrix,

$$Z_{ik} = - Z_{(i-1)(k-1)}/x.$$

Theorem 4

In the column direction of the Z matrix,

$$Z_{ik} = yZ_{(i+1)k} - xZ_{(i+2)k}.$$

Theorem 5

From Theorems 2, 3, and 4, the elements of the Z matrix are known if the elements of any row or of any column in the matrix are known.

Theorem 6

The Z matrix of order n-2 can be retrogressively derived from the Z matrix of order n by deleting the last two rows and columns in the matrix array of order n and dividing the residual elements by x^2.

Theorem 7

The Z matrix of order n+2 can be progressively constructed from the Z matrix of order n by multiplying the elements in the matrix array of order n by x^2 and applying theorems 3 and 4 to derive the elements in the additional two rows and columns.

From these theorems it is evident that when the elements of an $n \times n$ Z matrix are known, then the elements of all lower even-order Z matrices are immediately available. For example, the elements in the square area marked by $Z_{22} \ldots Z_{2n}$ and $Z_{22} \ldots Z_{n2}$ in the $n \times n$ matrix are exactly the same as those in the square area marked by $Z_{00} \ldots Z_{0(n-2)}$ and $Z_{00} \ldots Z_{(n-2)0}$ in the $(n-2) \times (n-2)$ matrix.*

*The author is obliged to John Ramsey, Lt, USAF, who noticed this identification during a diligent reading of the first draft of this report.

Likewise, the elements in the square area marked by $Z_{44} \ldots Z_{4n}$ and $Z_{44} \ldots Z_{n4}$ in the n×n matrix are exactly the same as those in the square area marked by $Z_{22} \ldots Z_{2(n-2)}$ and $Z_{22} \ldots Z_{(n-2)2}$ in the (n-2)× (n-2) matrix, as well as those in the square area marked by $Z_{00} \ldots Z_{0(n-4)}$ and $Z_{00} \ldots Z_{(n-4)0}$ in the (n-4)× (n-4) matrix. Obviously, this congruence provides a more convenient method than dividing the proper elements by x^2 or x^4 and so on.

7. REARRANGEMENT OF EQUATIONS DERIVED FROM COEFFICIENT COMPARISONS

7.1 Two Rearrangements of the Y Matrix Equations Derived From Comparing Coefficients

In Sec. 3.1 we compared coefficients and derived a system of n+1 equations in which the real coefficients $A_0, \ldots, A_n$ had the leading role. We can rearrange these equations so that either the unknown x or the unknown y has the leading role. We prefer to choose x for the part. As an example, let us rewrite Eqs. (3.5a) through (3.5e):

$$x^3A_3 - x^2(yA_2+A_1) + x(y^2A_1+2yA_0) - y^3A_0 \equiv 0 \quad (7.1a)$$

$$x^3 - x^2A_2 + x(yA_1+A_0) - y^2A_0 \equiv 0. \quad (7.1b)$$

$$x^2(y-A_3) + xA_1 - yA_0 \equiv 0. \quad (7.1c)$$

$$-x^2 + x(y^2-yA_3+A_2) - A_0 \equiv 0. \quad (7.1d)$$

$$- x(2y-A_3) + (y^3-y^2A_3+yA_2-A_1) \equiv 0. \quad (7.1e)$$

Except for formal multiplication by -1, Eqs. (7.1a) through (7.1e) are the same as Eqs. (3.5a) through (3.5e). In matrix form they are written

$$\left[Y_{ik}x^{4-k}\right]_{4,4} = \left[Y_{ik}\right]_{4,4} \cdot \left[x^{4-k}\right]_4 \equiv 0. \quad (7.1)$$

Equation (7.1) contains the Y matrix. This is a square matrix with 5×5 elements Y_{ik}. These elements are polynomials in y only. The coefficients of these Y_{ik} polynomials are the original coefficients $A_0, \ldots, A_4$, and thus real.

If, instead of x, we had chosen the unknown y, the equation would have contained the X matrix, written

$$\left[X_{ik}\, y^{4-k}\right]_{4,\,4} = \left[X_{ik}\right]_{4,\,4} \cdot \left[y^{4-k}\right]_{4} \equiv 0. \qquad (7.2)$$

A counterpart of Eq. (7.1) for the Y matrix, its development is therefore alternative and will be omitted.

7.2 Structure of the Y Matrix

Since the theorems presented in Sec. 6 enable us to find the Z matrix of any order it may seem superfluous to devote more than casual attention to the Y matrix. The Y matrix, as we have seen, results only from rewriting equations already known. Its structure, however, is such that its characteristic properties enable us fo find all the elements for any order without reference to any of its adjacent matrices. Because it deals with polynomials in only one unknown, y, it is also more amenable to solution than the Z matrix with its simultaneous polynomials in x and y.

The Y matrix of order 4, our first example showing the properties of its structure, has the extended form:

$$\begin{bmatrix} Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} \\ Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{20} & Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{30} & Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{40} & Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} = \left[Y_{ik}\right]_{4,\,4} \qquad (7.3)$$

From Eqs. (7.1a) through (7.1e), (7.1), and (7.3), we see that the following elements of the Y matrix are $\equiv 0$:

$$\left.\begin{aligned} Y_{00} = Y_{10} = Y_{20} = Y_{30} = Y_{40} &\equiv 0 \\ Y_{21} = Y_{31} = Y_{41} &\equiv 0 \\ Y_{42} &\equiv 0 \end{aligned}\right\} . \qquad (7.4)$$

Examining the Y matrix of any order, we find that, in general,

$$\left.\begin{array}{lll} Y_{i0} & \equiv 0 & (0 \leqq i \leqq n) \\ Y_{i1} & \equiv 0 & (2 \leqq i \leqq n) \\ Y_{i2} & \equiv 0 & (4 \leqq i \leqq n) \\ & \vdots & \\ Y_{n(n/2)} & \equiv 0 & \end{array}\right\} . \tag{7.5}$$

Let us now take a closer look at the elements in the main diagonal. In the matrix $\left[Y_{ik}\right]_{4,4}$ they are:

$Y_{00} \equiv 0$, as we know from Eq. (7.4).

$Y_{11} = A_4 \equiv 1$. Also, $Y_{11} = yY_{00} + A_4$.

$Y_{22} = y - A_3$. Also, $Y_{22} = yY_{11} - A_3$.

$Y_{33} = y^2 - yA_3 + A_2$. Also, $Y_{33} = yY_{22} + A_2$.

$Y_{44} = y^3 - y^2A_3 + yA_2 - A_1$. Also, $Y_{44} = yY_{33} - A_1$.

We see that in the Y matrix of any order, the elements of the main diagonal are

$$Y_{ii} = yY_{(i-1)(i-1)} - A_{n+1-i}\,(-1)^{n-i}, \tag{7.6}$$

where $0 \leqq i \leqq n$. Thus, the elements of the main diagonal of the Y matrix can be found immediately for any order n by an iterative procedure.

Let us now consider the elements arranged in the upper parallel to the main diagonal. This parallel, the co-diagonal, connects the element Y_{01} with the element Y_{34} in the Y matrix of order 4. Its elements are:

$Y_{01} = A_3, \quad Y_{12} = -A_2, \quad Y_{23} = A_1, \quad Y_{34} = -A_0$.

We see that in the Y matrix of any order, the elements of the co-diagonal are the coefficients of the polynomial f(z), those with an even index being opposite in polarity and those with an odd index being the same in polarity, as given by

$$Y_{i(i+1)} = + A_{n-i-1} (-1)^{n-i}, \tag{7.7}$$

where $0 \leqq i \leqq n-1$. Thus, the elements of the co-diagonal of the Y matrix are immediately known by the coefficients of the polynomial f(z).

Investigating the structure of the Y matrix in the column direction, we first consider the area below the main diagonal and then the area above the co-diagonal. Since both areas of the Y matrix of order 4 are too small to make the structure evident, we will use the Y matrix of order 6. Its elements are listed in Table 4 and known, since we know the elements of the Z matrix of order 6.

The elements of the main diagonal of the Y matrix of order 6 are, from Table 4,

$$\begin{aligned} &Y_{00} \equiv 0, && Y_{44} = y^3 - y^2 A_5 + yA_4 - A_3 \\ &Y_{11} = A_6 = 1, && Y_{55} = y^4 - y^3 A_5 + y^2 A_4 - yA_3 + A_2 \\ &Y_{22} = y - A_5, && Y_{66} = y^5 - y^4 A_5 + y^3 A_4 - y^2 A_3 + yA_2 - A_1 \\ &Y_{33} = y^2 - yA_5 + A_4 . && \end{aligned}$$

From this we see that the formula

$$Y_{ii} = \sum_{\mu=0}^{i-1} y^{i-(\mu+1)} A_{n-\mu} (-1)^{\mu} \tag{7.6a}$$

is more practical than Eq. (7.6) for finding the elements in the main diagonal.

The elements of the co-diagonal of the Y matrix of order 6 are, from Table 4,

$$\begin{aligned} &Y_{01} = A_5, && Y_{12} = -A_4, \\ &Y_{23} = A_3, && Y_{34} = -A_2, \\ &Y_{45} = A_1, && Y_{56} = -A_0. \end{aligned}$$

This result of course agrees with Eq. (7.7).

In the area below the main diagonal the elements below and including Y_{00}, Y_{21}, Y_{42}, and Y_{63} are identical with zero, as we know from Eq. (7.4). The elements still to be determined below the main diagonal are therefore one each in Columns 2 and 5 and two each in Columns 3 and 4, as the following array shows.

$$
\begin{array}{ccccccc|l}
0 & Y_{01} & Y_{02} & Y_{03} & Y_{04} & Y_{05} & Y_{06} & \\
0 & Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & \\
0 & 0 & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26} & \\
0 & 0 & Y_{32} & Y_{33} & Y_{34} & Y_{35} & Y_{36} & \\
0 & 0 & 0 & Y_{43} & Y_{44} & Y_{45} & Y_{46} & \\
0 & 0 & 0 & Y_{53} & Y_{54} & Y_{55} & Y_{56} & \text{co-diagonal} \\
0 & 0 & 0 & 0 & Y_{64} & Y_{65} & Y_{66} & \text{main diagonal}
\end{array}
$$

The elements not identical with zero in the diagonal immediately below the main diagonal are obtained by partial differentiation.

$$Y_{32} = -\frac{1}{1} \cdot \frac{\partial Y_{22}}{\partial y}, \quad Y_{43} = -\frac{1}{1} \cdot \frac{\partial Y_{33}}{\partial y},$$

$$Y_{54} = -\frac{1}{1} \cdot \frac{\partial Y_{44}}{\partial y}, \quad Y_{65} = -\frac{1}{1} \cdot \frac{\partial Y_{55}}{\partial y}.$$

The fraction -1/1 is explained when we consider such elements in the next lower diagonal.

$$Y_{42} = -\frac{1}{2} \frac{\partial Y_{32}}{\partial y} \equiv 0, \quad Y_{53} = -\frac{1}{2} \frac{\partial Y_{43}}{\partial y},$$

$$Y_{64} = -\frac{1}{2} \frac{\partial Y_{54}}{\partial y}.$$

Finally,

$$Y_{63} = -\frac{1}{3} \frac{\partial Y_{53}}{\partial y} = 0$$

fits in the same structure. It is apparent that the successive partial differentiation yields the statements in Eqs. (7.4) and thus makes those equations superfluous.

It would of course be premature to draw general conclusions from the results obtained for order 6 alone. The results obtained for the Y matrix of orders higher than 6 are, however, consistent in showing that starting from the element Y_{ii} in the main diagonal, where i=0,

$$\left.\begin{aligned} Y_{(i+1)i} &= -\frac{1}{1}\frac{\partial Y_{ii}}{\partial y}, \quad Y_{(i+2)i} = -\frac{1}{2}\frac{\partial Y_{(i+1)i}}{\partial y}, \\ Y_{(i+3)i} &= -\frac{1}{3}\frac{\partial Y_{(i+2)i}}{\partial y}, \ldots, Y_{(i+\rho)i} = -\frac{1}{\rho}\frac{\partial Y_{(i+\rho+1)i}}{\partial y} \end{aligned}\right\} \qquad (7.8)$$

until the partial differentiation yields zero for $\rho = i$. Hence, it can be stated that once the elements in the main diagonal are known, all the elements in the area below this partition can be obtained from Eq. (7.8).

Before we investigate the <u>area above the co-diagonal</u>, let us digress for a moment. Table 4 lists the elements $Y_{01}, \ldots Y_{06}$, in Row 0 of the Y matrix of order 6. The elements above the main diagonal in Row 1 are:

$$Y_{12} = +\frac{1}{1}\frac{\partial Y_{01}}{\partial y}, \quad Y_{13} = +\frac{1}{2}\frac{\partial Y_{03}}{\partial y}, \quad Y_{14} = +\frac{1}{3}\frac{\partial Y_{04}}{\partial y},$$

$$Y_{15} = +\frac{1}{4}\frac{\partial Y_{05}}{\partial y}, \quad Y_{16} = +\frac{1}{5}\frac{\partial Y_{06}}{\partial y},$$

Y_{12} being an element of the co-diagonal. The elements above the main diagonal in Row 2 are:

$$Y_{23} = +\frac{1}{1}\frac{\partial Y_{13}}{\partial y}, \quad Y_{24} = +\frac{1}{2}\frac{\partial Y_{14}}{\partial y}, \quad Y_{25} = +\frac{1}{3}\frac{\partial Y_{15}}{\partial y},$$

$$Y_{26} = +\frac{1}{4}\frac{\partial Y_{16}}{\partial y},$$

Y_{23} being an element of the co-diagonal. The elements above the main diagonal in Row 3 are:

$$Y_{34} = +\frac{1}{1}\frac{\partial Y_{24}}{\partial y}, \quad Y_{35} = +\frac{1}{2}\frac{\partial Y_{25}}{\partial y}, \quad Y_{36} = +\frac{1}{3}\frac{\partial Y_{26}}{\partial y},$$

Y_{34} being an element of the co-diagonal.

The elements above the main diagonal in Row 4 are

$$Y_{45} = +\frac{1}{1}\frac{\partial Y_{35}}{\partial y}, \quad Y_{46} = +\frac{1}{2}\frac{\partial Y_{36}}{\partial y},$$

Y_{45} being an element of the co-diagonal. Element Y_{56}, above the main diagonal in Row 5, is the last element in the co-diagonal:

$$Y_{56} = +\frac{1}{1}\frac{\partial Y_{46}}{\partial y}.$$

This iterative partial differentiation starting with the elements in Row 0 will yield the elements above the co-diagonal in the Y matrix of any order. The elements of the co-diagonal cannot be thus obtained, but these are already known.

We can also find the elements above the co-diagonal by an iterative integration, although this procedure leaves the constant of integration in doubt. To show how it workes we will compile the results for the Y matrix of order 6. We proceed by starting with each element in the co-diagonal, going up the columns by integration.

Since $Y_{00} \equiv 0$, and Y_{01} is an element of the co-diagonal, the element Y_{02} is the first that has to be found in the Y matrix of any order.

<u>Column 2</u>

$Y_{12} = -A_4$ is an element of the co-diagonal,

$$Y_{02} = -yA_4 - A_3 \quad \text{(see Table 4)}$$
$$= 1\int Y_{12}\,\partial y - A_3.$$

<u>Column 3</u>

$Y_{23} = A_3$ is an element of the co-diagonal,

$$Y_{13} = yA_3 + A_2 \quad \text{(see Table 4)}$$
$$= 1\int Y_{23}\,\partial y + A_2,$$
$$Y_{03} = y^2A_3 + 2yA_2 + A_1 \quad \text{(see Table 4)}$$
$$= 2\int Y_{13}\partial y + A_1.$$

Column 4

$$Y_{34} = -A_2,$$
$$Y_{24} = -yA_2 - A_1 = 1\int Y_{34}\partial y - A_1,$$
$$Y_{14} = -y^2A_2 - 2yA_1 - A_0 = 2\int Y_{24}\partial y - A_0,$$
$$Y_{04} = -y^3A_2 - 3y^2A_1 - 3yA_o = 3\int Y_{14}\partial y - 0.$$

Column 5

$$Y_{45} = A_1.$$
$$Y_{35} = yA_1 + A_0 = 1\int Y_{45}\partial y + A_0,$$
$$Y_{25} = y^2A_1 + 2yA_0 = 2\int Y_{35}\partial y + 0,$$
$$Y_{15} = y^3A_1 + 3y^2A_0 = 3\int Y_{25}\partial y + 0,$$
$$Y_{05} = y^4A_1 + 4y^3A_0 = 4\int Y_{15}\partial y + 0.$$

Column 6

$$Y_{56} = -A_0.$$
$$Y_{46} = -yA_0 = 1\int Y_{56}\partial y - 0,$$
$$Y_{36} = -y^2A_0 = 2\int Y_{46}\partial y - 0,$$
$$Y_{26} = -y^3A_0 = 3\int Y_{36}\partial y - 0,$$
$$Y_{16} = -y^4A_0 = 4\int Y_{26}\partial y - 0,$$
$$Y_{06} = -y^5A_0 = 5\int Y_{16}\partial y - 0.$$

The procedure used to achieve these results, which hold for the Y matrix of any order, is synthesized from the following rules. The next element in ascending order from the co-diagonal in any column is obtained from its preceding one by an integration multiplied by the sequence of integers 1, 2, 3, until Row 0 is reached. The descending sequence of polynomial coefficients, excluding the coefficient of the co-diagonal element, provides the constants of integration. The sequence being exhausted with A_0, the remaining constants of integration are taken as zero. The polarity of the constant of integration is determined by the polarity of the polynomial coefficient appearing in the co-diagonal: in the odd-numbered columns, the sign is the same; in the even-numbered columns, the sign is opposite.

More generally stated, these rules for obtaining all the elements above the co-diagonal are mathematically expressed in the iterative procedure of integration given by the following.

$$Y_{i(i+1)} = A_{n-(i+1)}(-1)^{n-i}$$

is an element of the co-diagonal known from Eq. (7.7). The elements in Column (i+1) with additional integration constants are:

$$\begin{aligned}
Y_{(i-1)(i+1)} &= 1\int Y_{i(i+1)}\,\partial y + A_{n-i-2}(-1)^{n-i},\\
Y_{(i-2)(i+1)} &= 2\int Y_{(i-1)(i+1)}\,\partial y + A_{n-i-3}(-1)^{n-i},\\
Y_{(i-3)(i+1)} &= 3\int Y_{(i-2)(i+1)}\,\partial y + A_{n-i-4}(-1)^{n-i},\\
&\vdots\\
Y_{(2i+1-n)(i+1)} &= (n-i-1)\int Y_{(2i+2-n)(i+1)}\,\partial y + A_0(-1)^{n-i}.
\end{aligned}$$

From here on the integration constants are zero.

$$\begin{aligned}
Y_{(2i-n)(i+1)} &= (n-i)\int Y_{(2i+1-n)(i+1)}\,\partial y + 0,\\
Y_{(2i-n-1)(i+1)} &= (n-i+1)\int Y_{(2i-n)(i+1)}\,\partial y + 0,\\
&\vdots\\
Y_{0(i+1)} &= i\int Y_{1(i+1)}\,\partial y + 0.
\end{aligned}$$

Here we reach Row 0.

Thus, an element in the area above the co-diagonal is

$$Y_{(i-\mu)(i+1)} = \mu\int Y_{(i-\mu+1)}\partial y + A_{n-i-\mu+1}(-1)^{n-i} \qquad (7.9\underline{a})$$

from Row i-1(μ=1) to Row 2i-n+1(μ=n-i-1) inclusive,

and

$$Y_{(i-\mu)(i+1)} = \mu\int Y_{(i-\mu+1)}\partial y + 0 \qquad (7.9\underline{b})$$

from Row 2(i-1)-n(μ=n-i) to Row 0(μ=i).

The progressive integration is less complicated than it appears. The computation is very easy if we start with the last column n in the Y matrix.

To find the elements in a particular row of the Y matrix we can of course skip some of the steps in the partial differentiation, as well as in the integration, and still get any of Eqs. (7.1). This bears out the earlier statement that any of Eqs. (3.6) can be obtained immediately.

8. THE Y MATRIX THEOREMS

Theorem 8

The n+1 equations derived from comparing the coefficients of a polynomial of even order n with real coefficients can be rearranged to derive a matrix with elements Y_{ik} that are polynomials in one unknown, y. The matrix is called the Y matrix $\left[Y_{ik}\right]_{n,n}$.

Theorem 9

Beginning with the diagonal element $Y_{00} \equiv 0$, the elements of the main diagonal of the Y matrix can be obtained iteratively from

$$Y_{ii} = y\,Y_{(i-1)(i-1)} - A_{n-i+1}(-1)^{n-i}.$$

Theorem 10

The co-diagonal of the Y matrix is defined as the parallel to the main diagonal comprising the elements Y_{01}, Y_{12}, ..., $Y_{(n-1)n}$. Its elements are given by

$$Y_{i(i+1)} = A_{n-i-1}(-1)^{n-i}.$$

Theorem 11

The elements below the main diagonal of the Y matrix can be obtained by a process of successive partial differentiation from one element to the next one within each column according to Eqs. (7.8).

Theorem 12

The elements above the co-diagonal of the Y matrix can be obtained by a process of successive integration from one element to the next one within each column according to Eqs. (7.9a) and (7.9b).

9. A NOTE ABOUT THE APPENDIX

The appendix contains workcheck tabulations of the Z and Y matrices and their elements for the even orders 2 through 12. Although the report is complete without them, they are provided for the reader's convenience in following the practical demonstrations. Comparisons of the matrix elements within a particular order with those of other orders will disclose many properties other than those covered by the 12 theorems. The tabulations 2-Y to 12-Y, for instance, show that the elements in the last column of the Y matrix, except the element in the last row, represent a geometric progression whose common ratio is y and first term A_0. All such additional properties can be elicited as consequences of the theorems, which are sufficient to obtain the matrix of any order.

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TABLE 1. Comparison of Coefficients of Orders 2, 4, and 6

	ORDER 2	ORDER 4	ORDER 6
A_0	x_1	x_1x_2	$x_1x_2x_3$
A_1	y_1	$y_1x_2+y_2x_1$	$y_1x_2x_3+y_2x_1x_3+y_3x_1x_2$
A_2	$\equiv 1$	$x_1+x_2+y_1y_2$	$x_1x_2+x_1x_3+x_2x_3+x_1y_2y_3+x_2y_1y_3+x_3y_1y_2$
A_3		y_1+y_2	$y_1(x_2+x_3)+y_2(x_1+x_3)+y_3(x_1+x_2)+y_1y_2y_3$
A_4		$\equiv 1$	$x_1+x_2+x_3+y_1y_2+y_1y_3+y_2y_3$
A_5			$y_1+y_2+y_3$
A_6			$\equiv 1$

TABLE 2. Elements of the Matrix $\left[Z_{ik}\right]_{n,n}$ of Orders 2, 4, and 6

	ORDER 2	ORDER 4	ORDER 6		ORDER 4	ORDER 6
$Z_{01} = Z_{10}$	$-x$	$-x^3$	$-x^5$	$Z_{23} = Z_{32}$	$-x$	$-x^3$
$Z_{02} = -Z_{20}$	y	x^2y	x^4y	$Z_{24} = -Z_{42}$	y	x^2y
$Z_{03} = Z_{30}$		$-x(y^2-x)$	$-x^3(y^2-x)$	$Z_{25} = Z_{52}$		$-x(y^2-x)$
$Z_{04} = -Z_{40}$		$y(y^2-2x)$	$x^2y(y^2-2x)$	$Z_{26} = -Z_{62}$		$y(y^2-2x)$
$Z_{05} = Z_{50}$			$-x(y^4-3xy^2+x^2)$	$Z_{34} = Z_{43}$	$+1$	x^2
$Z_{06} = -Z_{60}$			$y(y^4-4xy^2+3x^2)$	$Z_{35} = -Z_{53}$		$-xy$
$Z_{12} = Z_{21}$	$+1$	x^2	x^4	$Z_{36} = Z_{63}$		y^2-x
$Z_{13} = -Z_{31}$		$-xy$	$-x^3y$	$Z_{45} = Z_{54}$		$-x$
$Z_{14} = Z_{41}$		y^2-x	$x^2(y^2-x)$	$Z_{46} = -Z_{64}$		y
$Z_{15} = -Z_{51}$			$-xy(y^2-2x)$	$Z_{56} = Z_{65}$		$+1$
$Z_{16} = Z_{61}$			$y^4-3xy^2+x^2$			

TABLE 3. Elements of Row 0 of the Z Matrix of Order 12

Z_{00}	0
$Z_{01} = Z_{10}$	$-x^{11}$
$Z_{02} = -Z_{20}$	$x^{10}y$
$Z_{03} = Z_{30}$	$-x^9(y^2-x)$
$Z_{04} = -Z_{40}$	$x^8y(y^2-2x)$
$Z_{05} = Z_{50}$	$-x^7(y^4-3xy^2+x^2)$
$Z_{06} = -Z_{60}$	$x^6y(y^4-4xy^2+3x^2)$
$Z_{07} = Z_{70}$	$-x^5(y^6-5xy^4+6x^2y^2-x^3)$
$Z_{08} = -Z_{80}$	$x^4y(y^6-6xy^4+10x^2y^2-4x^3)$
$Z_{09} = Z_{90}$	$-x^3(y^8-7xy^6+15x^2y^4-10x^3y^2+x^4)$
$Z_{0'10} = -Z_{10'0}$	$x^2y(y^8-8xy^6+21x^2y^4-20x^3y^2+5x^4)$
$Z_{0'11} = Z_{11'0}$	$-x(y^{10}-9xy^8+28x^2y^6-35x^3y^4+15x^4y^2-x^5)$
$Z_{0'12} = -Z_{12'0}$	$y(y^{10}-10xy^8+36x^2y^6-56x^3y^4+35x^4y^2-6x^5)$

TABLE 4. Elements Not Identical With Zero in the Y Matrix of Order 6

$Y_{01} = A_5$ $Y_{02} = -(yA_4+A_3)$ $Y_{03} = y^2A_3+2yA_2+A_1$ $Y_{04} = -y(y^2A_2+3yA_1+3A_0)$ $Y_{05} = y^3(yA_1+4A_0)$ $Y_{06} = -y^5A_0$	$Y_{11} = +1$ $Y_{12} = -A_4$ $Y_{13} = yA_3+A_2$ $Y_{14} = -(y^2A_2+2yA_1+A_0)$ $Y_{15} = y^2(yA_1+3A_0)$ $Y_{16} = -y^4A_0$
$Y_{22} = y-A_5$ $Y_{23} = A_3$ $Y_{24} = -(yA_2+A_1)$ $Y_{25} = y(yA_1+2A_0)$ $Y_{26} = -y^3A_0$	$Y_{32} = -1$ $Y_{33} = y^2-yA_5+A_4$ $Y_{34} = -A_2$ $Y_{35} = yA_1+A_0$ $Y_{36} = -y^2A_0$
$Y_{43} = -(2y-A_5)$ $Y_{44} = y^3-y^2A_5+yA_4-A_3$ $Y_{45} = A_1$ $Y_{46} = -yA_0$	$Y_{53} = +1$ $Y_{54} = -(3y^2-2yA_5+A_4)$ $Y_{55} = y^4-y^3A_5+y^2A_4-yA_3+A_2$ $Y_{56} = -A_0$
$Y_{64} = 3y-A_5$ $Y_{65} = -(4y^3-3y^2A_5+2yA_4-A_3)$ $Y_{66} = y^5-y^4A_5+y^3A_4-y^2A_3+yA_2-A_1$	

APPENDIX

ORDER 2

Polynomial $f(z) = z^2 + A_1 z + A_0$

Z-Matrix (elements see TABULATION 2-Z)

$$\begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix}$$

Y-Matrix (elements see TABULATION 2-Y)

$$\begin{bmatrix} Y_{00} & Y_{01} & Y_{02} \\ 0 & Y_{11} & Y_{12} \\ 0 & 0 & Y_{22} \end{bmatrix}$$

TABULATION 2-Z

ROW 0

$Z_{01} = Z_{10} = -x$

$Z_{02} = -Z_{20} = y$

ROW 1

$Z_{12} = Z_{21} = +1$

TABULATION 2-Y

ROW 0

$Y_{00} = 0$

$Y_{01} = A_1$

$Y_{02} = -yA_0$

ROW 1

$Y_{11} = +1$

$Y_{12} = -A_0$

ROW 2

$Y_{22} = y - A_1$

ORDER 4

Polynomial $f(z) = z^4 + A_3 z^3 + \cdots + A_1 z + A_0$

Z- Matrix (elements see TABULATION 4-Z)

$$\begin{bmatrix} z_{00} & z_{01} & z_{02} & z_{03} & z_{04} \\ z_{10} & z_{11} & z_{12} & z_{13} & z_{14} \\ z_{20} & z_{21} & z_{22} & z_{23} & z_{24} \\ z_{30} & z_{31} & z_{32} & z_{33} & z_{34} \\ z_{40} & z_{41} & z_{42} & z_{43} & z_{44} \end{bmatrix}$$

Y- Matrix (elements see TABULATION 4-Y)

$$\begin{bmatrix} Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} \\ 0 & Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ 0 & 0 & Y_{22} & Y_{23} & Y_{24} \\ 0 & 0 & Y_{32} & Y_{33} & Y_{34} \\ 0 & 0 & 0 & Y_{43} & Y_{44} \end{bmatrix}$$

TABULATION 4-Z

ROW 0

$Z_{01} = Z_{10} = -x^3$

$Z_{02} = -Z_{20} = x^2y$

$Z_{03} = Z_{30} = -x(y^2 - x)$

$Z_{04} = -Z_{40} = y(y^2 - 2x)$

ROW 1

$Z_{12} = Z_{21} = x^2$

$Z_{13} = -Z_{31} = -xy$

$Z_{14} = Z_{41} = y^2 - x$

ROW 2

$Z_{23} = Z_{32} = -x$

$Z_{24} = -Z_{42} = y$

ROW 3

$Z_{34} = Z_{43} = +1$

TABULATION 4-Y

ROW 0

$Y_{00} = 0$

$Y_{01} = A_3$

$Y_{02} = -(yA_2 + A_1)$

$Y_{03} = y(yA_1 + 2A_0)$

$Y_{04} = -y^3A_0$

ROW 1

$Y_{11} = +1$

$Y_{12} = -A_2$

$Y_{13} = yA_1 + A_0$

$Y_{14} = -y^2A_0$

ROW 2

$Y_{22} = y - A_3$

$Y_{23} = A_1$

$Y_{24} = -yA_0$

ROW 3

$Y_{32} = -1$

$Y_{33} = y^2 - yA_3 + A_2$

$Y_{34} = -A_0$

ROW 4

$Y_{43} = -(2y - A_3)$

$Y_{44} = y^3 - y^2A_3 + yA_2 - A_1$

ORDER 6

Polynomial $f(z) = z^6 + A_5 z^5 + \cdots + A_1 z + A_0$

Z-Matrix (elements see TABULATION 6-Z)

$$
\begin{bmatrix}
Z_{00} & Z_{01} & Z_{02} & Z_{03} & Z_{04} & Z_{05} & Z_{06} \\
Z_{10} & Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} \\
Z_{20} & Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} \\
Z_{30} & Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} & Z_{36} \\
Z_{40} & Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & Z_{46} \\
Z_{50} & Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} \\
Z_{60} & Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66}
\end{bmatrix}
$$

Y-Matrix (elements see TABULATION 6-Y)

$$
\begin{bmatrix}
Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} & Y_{05} & Y_{06} \\
0 & Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} \\
0 & 0 & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26} \\
0 & 0 & Y_{32} & Y_{33} & Y_{34} & Y_{35} & Y_{36} \\
0 & 0 & 0 & Y_{43} & Y_{44} & Y_{45} & Y_{46} \\
0 & 0 & 0 & Y_{53} & Y_{54} & Y_{55} & Y_{56} \\
0 & 0 & 0 & 0 & Y_{64} & Y_{65} & Y_{66}
\end{bmatrix}
$$

TABULATION 6-Z

ROW 0

$Z_{01} = Z_{10} = -x^5$

$Z_{02} = -Z_{20} = x^4y$

$Z_{03} = Z_{30} = -x^3(y^2 - x)$

$Z_{04} = -Z_{40} = x^2y(y^2 - 2x)$

$Z_{05} = Z_{50} = -x(x^2 - 3xy^2 + y^4)$

$Z_{06} = -Z_{60} = y(3x^2 - 4xy^2 + y^4)$

ROW 1

$Z_{12} = Z_{21} = x^4$

$Z_{13} = -Z_{31} = -x^3y$

$Z_{14} = Z_{41} = x^2(y^2 - x)$

$Z_{15} = -Z_{51} = -xy(y^2 - 2x)$

$Z_{16} = Z_{61} = x^2 - 3xy^2 + y^4$

ROW 2

$Z_{23} = Z_{32} = -x^3$

$Z_{24} = -Z_{42} = x^2y$

$Z_{25} = Z_{52} = -x(y^2 - x)$

$Z_{26} = -Z_{62} = y(y^2 - 2x)$

ROW 3

$Z_{34} = Z_{43} = x^2$

$Z_{35} = -Z_{53} = -xy$

$Z_{36} = Z_{63} = y^2 - x$

ROW 4

$Z_{45} = Z_{54} = -x$

$Z_{46} = -Z_{64} = y$

ROW 5

$Z_{56} = Z_{65} = +1$

TABULATION 6-Y

ROW 0

$Y_{00} = 0$

$Y_{01} = A_5$

$Y_{02} = -(yA_4 + A_3)$

$Y_{03} = y^2A_3 + 2yA_2 + A_1$

$Y_{04} = -y(y^2A_2 + 3yA_1 + 3A_0)$

$Y_{05} = y^3(yA_1 + 4A_0)$

$Y_{06} = -y^5A_0$

ROW 1

$Y_{11} = +1$

$Y_{12} = -A_4$

$Y_{13} = yA_3 + A_2$

$Y_{14} = -(y^2A_2 + 2yA_1 + A_0)$

$Y_{15} = y^2(yA_1 + 3A_0)$

$Y_{16} = -y^4A_0$

ROW 2

$Y_{22} = y - A_5$

$Y_{23} = A_3$

$Y_{24} = -(yA_2 + A_1)$

$Y_{25} = y(yA_1 + 2A_0)$

$Y_{26} = -y^3A_0$

ROW 3

$Y_{32} = -1$

$Y_{33} = y^2 - yA_5 + A_4$

$Y_{34} = -A_2$

$Y_{35} = yA_1 + A_0$

$Y_{36} = -y^2A_0$

ROW 4

$Y_{43} = -(2y - A_5)$

$Y_{44} = y^3 - y^2A_5 + yA_4 - A_3$

$Y_{45} = A_1$

$Y_{46} = yA_0$

ROW 5

$Y_{53} = +1$

$Y_{54} = -(3y^2 - 2yA_5 + A_4)$

$Y_{55} = y^4 - y^3A_5 + y^2A_4 - yA_3 + A_2$

$Y_{56} = -A_0$

ROW 6

$Y_{64} = 3y - A_5$

$Y_{65} = -(4y^3 - 3y^2A_5 + 2yA_4 - A_3)$

$Y_{66} = y^5 - y^4A_5 + y^3A_4 - y^2A_3 + yA_2 - A_1$

ORDER 8

Polynomial $f(z) = z^8 + A_7 z^7 + \cdots + A_1 z + A_0$

Z-Matrix

(elements see TABULATION 8-Z)

$$\begin{bmatrix} Z_{00} & Z_{01} & Z_{02} & Z_{03} & Z_{04} & Z_{05} & Z_{06} & Z_{07} & Z_{08} \\ Z_{10} & Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} \\ Z_{20} & Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} & Z_{27} & Z_{28} \\ Z_{30} & Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} & Z_{36} & Z_{37} & Z_{38} \\ Z_{40} & Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & Z_{46} & Z_{47} & Z_{48} \\ Z_{50} & Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} & Z_{57} & Z_{58} \\ Z_{60} & Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66} & Z_{67} & Z_{68} \\ Z_{70} & Z_{71} & Z_{72} & Z_{73} & Z_{74} & Z_{75} & Z_{76} & Z_{77} & Z_{78} \\ Z_{80} & Z_{81} & Z_{82} & Z_{83} & Z_{84} & Z_{85} & Z_{86} & Z_{87} & Z_{88} \end{bmatrix}$$

Y-Matrix

(elements see TABULATION 8-Y)

$$\begin{bmatrix} Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} & Y_{05} & Y_{06} & Y_{07} & Y_{08} \\ 0 & Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & Y_{17} & Y_{18} \\ 0 & 0 & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26} & Y_{27} & Y_{28} \\ 0 & 0 & Y_{32} & Y_{33} & Y_{34} & Y_{35} & Y_{36} & Y_{37} & Y_{38} \\ 0 & 0 & 0 & Y_{43} & Y_{44} & Y_{45} & Y_{46} & Y_{47} & Y_{48} \\ 0 & 0 & 0 & Y_{53} & Y_{54} & Y_{55} & Y_{56} & Y_{57} & Y_{58} \\ 0 & 0 & 0 & 0 & Y_{64} & Y_{65} & Y_{66} & Y_{67} & Y_{68} \\ 0 & 0 & 0 & 0 & Y_{74} & Y_{75} & Y_{76} & Y_{77} & Y_{78} \\ 0 & 0 & 0 & 0 & 0 & Y_{85} & Y_{86} & Y_{87} & Y_{88} \end{bmatrix}$$

TABULATION 8-Z

ROW 0

$$z_{01} = z_{10} = -x^7$$

$$z_{02} = -z_{20} = x^6 y$$

$$z_{03} = z_{30} = -x^5(y^2 - x)$$

$$z_{04} = -z_{40} = x^4 y(y^2 - 2x)$$

$$z_{05} = z_{50} = -x^3(x^2 - 3xy^2 + y^4)$$

$$z_{06} = -z_{60} = x^2 y(3x^2 - 4xy^2 + y^4)$$

$$z_{07} = z_{70} = -x(y^6 - 5xy^4 + 6x^2y^2 - x^3)$$

$$z_{08} = -z_{80} = y(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$$

ROW 1

$$z_{12} = z_{21} = x^6$$

$$z_{13} = -z_{31} = -x^5 y$$

$$z_{14} = z_{41} = x^4(y^2 - x)$$

$$z_{15} = -z_{51} = -x^3 y(y^2 - 2x)$$

$$z_{16} = z_{61} = x^2(x^2 - 3xy^2 + y^4)$$

$$z_{17} = -z_{71} = -xy(3x^2 - 4xy^2 + y^4)$$

$$z_{18} = z_{81} = y^6 - 5xy^4 + 6x^2y^2 - x^3$$

ROW 2

$$z_{23} = z_{32} = -x^5$$

$$z_{24} = -z_{42} = x^4 y$$

$$z_{25} = z_{52} = -x^3(y^2 = x)$$

$$z_{26} = -z_{62} = x^2 y(y^2 - 2x)$$

$$z_{27} = z_{72} = -x(x^2 - 3xy^2 + y^4)$$

$$z_{28} = -z_{82} = y(3x^2 - 4xy^2 + y^4)$$

ROW 3

$$z_{34} = z_{43} = x^4$$

$$z_{35} = -z_{53} = -x^3 y$$

$$z_{36} = z_{63} = x^2(y^2 - x)$$

$$z_{37} = -z_{73} = -xy(y^2 - 2x)$$

$$z_{38} = z_{83} = x^2 - 3xy^2 + y^4$$

ROW 4

$$z_{45} = z_{54} = -x^3$$

$$z_{46} = -z_{64} = x^2 y$$

$$z_{47} = z_{74} = -x(y^2 - x)$$

$$z_{48} = -z_{84} = y(y^2 - 2x)$$

ROW 5

$$z_{56} = z_{65} = x^2$$

$$z_{57} = -z_{75} = -xy$$

$$z_{58} = z_{85} = y^2 - x$$

ROW 6

$$z_{67} = z_{76} = -x$$

$$z_{68} = -z_{86} = y$$

ROW 7

$$z_{78} = z_{87} = +1$$

TABULATION 8-Y

ROW 0

$Y_{00} = 0$

$Y_{01} = A_7$

$Y_{02} = -(yA_6 + A_5)$

$Y_{03} = y^2A_5 + 2yA_4 + A_3$

$Y_{04} = -(y^3A_4 + 3y^2A_3 + 3yA_2 + A_1)$

$Y_{05} = y(y^3A_3 + 4y^2A_2 + 6yA_1 + 4A_0)$

$Y_{06} = -y^3(y^2A_2 + 5yA_1 + 10A_0)$

$Y_{07} = y^5(yA_1 + 6A_0)$

$Y_{08} = -y^7A_0$

ROW 1

$Y_{11} = +1$

$Y_{12} = -A_6$

$Y_{13} = yA_5 + A_4$

$Y_{14} = -(y^2A_4 + 2yA_3 + A_2)$

$Y_{15} = y^3A_3 + 3y^2A_2 + 3yA_1 + A_0$

$Y_{16} = -y^2(y^2A_2 + 4yA_1 + 6A_0)$

$Y_{17} = y^4(yA_1 + 5A_0)$

$Y_{18} = -y^6A_0$

ROW 2

$Y_{22} = y - A_7$

$Y_{23} = A_5$

$Y_{24} = -(yA_4 + A_3)$

$Y_{25} = y^2A_3 + 2yA_2 + A_1$

$Y_{26} = -y(y^2A_2 + 3yA_1 + 3A_0)$

$Y_{27} = y^3(yA_1 + 4A_0)$

$Y_{28} = -y^5A_0$

ROW 3

$Y_{32} = -1$

$Y_{33} = y^2 - yA_7 + A_6$

$Y_{34} = -A_4$

$Y_{35} = yA_3 + A_2$

$Y_{36} = -(y^2A_2 + 2yA_1 + A_0)$

$Y_{37} = y^2(yA_1 + 3A_0)$

$Y_{38} = -y^4A_0$

ROW 4

$Y_{43} = -(2y - A_7)$

$Y_{44} = y^3 - y^2A_7 + yA_6 - A_5$

$Y_{45} = A_3$

$Y_{46} = -(yA_2 + A_1)$

$Y_{47} = y(yA_1 + 2A_0)$

$Y_{48} = -y^3A_0$

ROW 5

$Y_{53} = +1$

$Y_{54} = -(3y^2 - 2yA_7 + A_6)$

$Y_{55} = y^4 - y^3A_7 + y^2A_6 - yA_5 + A_4$

$Y_{56} = -A_2$

$Y_{57} = yA_1 + A_0$

$Y_{58} = -y^2A_0$

ROW 6

$Y_{64} = 3y - A_7$

$Y_{65} = -(4y^3 - 3y^2A_7 + 2yA_6 - A_5)$

$Y_{66} = y^5 - y^4A_7 + y^3A_6 - y^2A_5 + yA_4 - A_3$

$Y_{67} = A_1$

$Y_{68} = -yA_0$

ROW 7

$Y_{74} = -1$

$Y_{75} = 6y^2 - 3yA_7 + A_6$

$Y_{76} = -(5y^4 - 4y^3A_7 + 3y^2A_6 - 2yA_5 + A_4)$

$Y_{77} = y^6 - y^5A_7 + y^4A_6 - y^3A_5 + y^2A_4 - yA_3 + A_2$

$Y_{78} = -A_0$

ROW 8

$Y_{85} = -(4y - A_7)$

$Y_{86} = 10y^3 - 6y^2A_7 + 3yA_6 - A_5$

$Y_{87} = -(6y^5 - 5y^4A_7 + 4y^3A_6 - 3y^2A_5 + 2yA_4 - A_3)$

$Y_{88} = y^7 - y^6A_7 + y^5A_6 - y^4A_5 + y^3A_4 - y^2A_3 + yA_2 - A_1$

ORDER 10

Polynomial $f(z) = z^{10} + A_9 z^9 + \cdots + A_1 z + A_0$

Z – Matrix (elements see TABULATION 10-Z)

$$
\begin{bmatrix}
z_{00} & z_{01} & z_{02} & z_{03} & z_{04} & z_{05} & z_{06} & z_{07} & z_{08} & z_{09} & z_{0(10)} \\
z_{10} & z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} & z_{17} & z_{18} & z_{19} & z_{1(10)} \\
z_{20} & z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} & z_{27} & z_{28} & z_{29} & z_{2(10)} \\
z_{30} & z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} & z_{37} & z_{38} & z_{39} & z_{3(10)} \\
z_{40} & z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} & z_{47} & z_{48} & z_{49} & z_{4(10)} \\
z_{50} & z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} & z_{57} & z_{58} & z_{59} & z_{5(10)} \\
z_{60} & z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} & z_{67} & z_{68} & z_{69} & z_{6(10)} \\
z_{70} & z_{71} & z_{72} & z_{73} & z_{74} & z_{75} & z_{76} & z_{77} & z_{78} & z_{79} & z_{7(10)} \\
z_{80} & z_{81} & z_{82} & z_{83} & z_{84} & z_{85} & z_{86} & z_{87} & z_{88} & z_{89} & z_{8(10)} \\
z_{90} & z_{91} & z_{92} & z_{93} & z_{94} & z_{95} & z_{96} & z_{97} & z_{98} & z_{99} & z_{9(10)} \\
z_{(10)0} & z_{(10)1} & z_{(10)2} & z_{(10)3} & z_{(10)4} & z_{(10)5} & z_{(10)6} & z_{(10)7} & z_{(10)8} & z_{(10)9} & z_{(10)(10)}
\end{bmatrix}
$$

Y – Matrix (elements see TABULATION 10-Y)

$$
\begin{bmatrix}
Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} & Y_{05} & Y_{06} & Y_{07} & Y_{08} & Y_{09} & Y_{0(10)} \\
0 & Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & Y_{17} & Y_{18} & Y_{19} & Y_{1(10)} \\
0 & 0 & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26} & Y_{27} & Y_{28} & Y_{29} & Y_{2(10)} \\
0 & 0 & Y_{32} & Y_{33} & Y_{34} & Y_{35} & Y_{36} & Y_{37} & Y_{38} & Y_{39} & Y_{3(10)} \\
0 & 0 & 0 & Y_{43} & Y_{44} & Y_{45} & Y_{46} & Y_{47} & Y_{48} & Y_{49} & Y_{4(10)} \\
0 & 0 & 0 & Y_{53} & Y_{54} & Y_{55} & Y_{56} & Y_{57} & Y_{58} & Y_{59} & Y_{5(10)} \\
0 & 0 & 0 & 0 & Y_{64} & Y_{65} & Y_{66} & Y_{67} & Y_{68} & Y_{69} & Y_{6(10)} \\
0 & 0 & 0 & 0 & Y_{74} & Y_{75} & Y_{76} & Y_{777} & Y_{78} & Y_{79} & Y_{7(10)} \\
0 & 0 & 0 & 0 & 0 & Y_{85} & Y_{86} & Y_{87} & Y_{88} & Y_{89} & Y_{8(10)} \\
0 & 0 & 0 & 0 & 0 & Y_{95} & Y_{96} & Y_{97} & Y_{98} & Y_{99} & Y_{9(10)} \\
0 & 0 & 0 & 0 & 0 & 0 & Y_{(10)6} & Y_{(10)7} & Y_{(10)8} & Y_{(10)9} & Y_{(10)(10)}
\end{bmatrix}
$$

TABULATION 10-Z

ROW 0

$$z_{01} = z_{10} = -x^9$$
$$z_{02} = -z_{20} = x^8y$$
$$z_{03} = z_{30} = -x^7(y^2 - x)$$
$$z_{04} = -z_{40} = x^6y(y^2 - 2x)$$
$$z_{05} = z_{50} = -x^5(x^2 - 3xy^2 + y^4)$$
$$z_{06} = -z_{60} = x^4y(3x^2 - 4xy^2 + y^4)$$
$$z_{07} = z_{70} = -x^3(y^6 - 5xy^4 + 6x^2y^2 - x^3)$$
$$z_{08} = -z_{80} = x^2y(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$$
$$z_{09} = z_{90} = -x(y^8 - 7xy^6 + 15x^2y^4 - 10x^3y^2 + x^4)$$
$$z_{0(10)} = -z_{0(10)} = y(y^8 - 8xy^6 + 21x^2y^4 - 20x^3y^2 + 5x^4)$$

ROW 1

$$z_{12} = z_{21} = x^8$$
$$z_{13} = -z_{31} = -x^7y$$
$$z_{14} = z_{41} = x^6(y^2 - x)$$
$$z_{15} = -z_{51} = -x^5y(y^2 - 2x)$$
$$z_{16} = z_{61} = x^4(x^2 - 3xy^2 + y^4)$$
$$z_{17} = -z_{71} = -x^3y(3x^2 - 4xy^2 + y^4)$$
$$z_{18} = z_{81} = x^2(y^6 - 5xy^4 + 6x^2y^2 - x^3)$$
$$z_{19} = -z_{91} = -xy(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$$
$$z_{1(10)} = z_{(10)1} = y^8 - 7xy^6 + 15x^2y^4 - 10x^3y^2 + x^4$$

ROW 2

$$z_{23} = z_{32} = -x^7$$
$$z_{24} = -z_{42} = x^6y$$
$$z_{25} = z_{52} = -x^5(y^2 - x)$$
$$z_{26} = -z_{62} = x^4y(y^2 - 2x)$$
$$z_{27} = z_{72} = -x^3(x^2 - 3xy^2 + y^4)$$
$$z_{28} = -z_{82} = x^2y(3x^2 - 4xy^2 + y^4)$$
$$z_{29} = z_{92} = -x(y^6 - 5xy^4 + 6x^2y^2 - x^3)$$
$$z_{2(10)} = -z_{(10)2} = y(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$$

ROW 3

$$z_{34} = z_{43} = x^6$$
$$z_{35} = -z_{53} = -x^5y$$
$$z_{36} = z_{63} = x^4(y^2 - x)$$
$$z_{37} = -z_{73} = -x^3y(y^2 - 2x)$$
$$z_{38} = z_{83} = x^2(x^2 - 3xy^2 + y^4)$$
$$z_{39} = -z_{93} = -xy(3x^2 - 4xy^2 + y^4)$$
$$z_{3(10)} = z_{(10)3} = y^6 - 5xy^4 + 6x^2y^2 - x^3$$

ROW 4

$$z_{45} = z_{54} = -x^5$$
$$z_{46} = -z_{64} = x^4y$$
$$z_{47} = z_{74} = -x^3(y^2 - x)$$
$$z_{48} = -z_{84} = x^2y(y^2 - 2x)$$
$$z_{49} = z_{94} = -x(x^2 - 3xy^2 + y^4)$$
$$z_{4(10)} = -z_{(10)4} = y(3x^2 - 4xy^2 + y^4)$$

ROW 5

$$z_{56} = z_{65} = x^4$$
$$z_{57} = -z_{75} = -x^3y$$
$$z_{58} = z_{85} = x^2(y^2 - x)$$
$$z_{59} = -z_{95} = -xy(y^2 - 2x)$$
$$z_{5(10)} = z_{(10)5} = x^2 - 3xy^2 + y^4$$

ROW 6

$$z_{67} = z_{76} = -x^3$$
$$z_{68} = -z_{86} = x^2y$$
$$z_{69} = z_{96} = -x(y^2 - x)$$
$$z_{6(10)} = -z_{(10)6} = y(y^2 - 2x)$$

ROW 7

$$z_{78} = z_{87} = x^2$$
$$z_{79} = -z_{97} = -xy$$
$$z_{7(10)} = z_{(10)7} = y^2 - x$$

ROW 8

$$z_{89} = z_{98} = -x$$
$$z_{8(10)} = -z_{(10)8} = y$$

ROW 9

$$z_{9(10)} = z_{(10)9} = +1$$

TABULATION 10-Y

ROW 0

$$Y_{00} = 0$$
$$Y_{01} = A_9$$
$$Y_{02} = -(yA_8 + A_7)$$
$$Y_{03} = y^2A_7 + 2yA_6 + A_5$$
$$Y_{04} = -(y^3A_6 + 3y^2A_5 + 3yA_4 + A_3)$$
$$Y_{05} = y^4A_5 + 4y^3A_4 + 6y^2A_3 + 4yA_2 + A_1$$
$$Y_{06} = -y(y^4A_4 + 5y^3A_3 + 10y^2A_2 + 10yA_1 + 5A_0)$$
$$Y_{07} = y^3(y^3A_3 + 6y^2A_2 + 15yA_1 + 20A_0)$$
$$Y_{08} = -y^5(y^2A_2 + 7yA_1 + 21A_0)$$
$$Y_{09} = y^7(yA_1 + 8A_0)$$
$$Y_{0(10)} = -y^9A_0$$

ROW 1

$$Y_{11} = +1$$
$$Y_{12} = -A_8$$
$$Y_{13} = yA_7 + A_6$$
$$Y_{14} = -(y^2A_6 + 2yA_5 + A_4)$$
$$Y_{15} = y^3A_5 + 3y^2A_4 + 3yA_3 + A_2$$
$$Y_{16} = -(y^4A_4 + 4y^3A_3 + 6y^2A_2 + 4yA_1 + A_0)$$
$$Y_{17} = y^2(y^3A_3 + 5y^2A_2 + 10yA_1 + 10A_0)$$
$$Y_{18} = -y^4(y^2A_2 + 6yA_1 + 15A_0)$$
$$Y_{19} = y^6(yA_1 + 7A_0)$$
$$Y_{1(10)} = -y^8A_0$$

ROW 2

$$Y_{22} = y - A_9$$
$$Y_{23} = A_7$$
$$Y_{24} = -(yA_6 + A_5)$$
$$Y_{25} = y^2A_5 + 2yA_4 + A_3$$
$$Y_{26} = -(y^3A_4 + 3y^2A_3 + 3yA_2 + A_1)$$
$$Y_{27} = y(y^3A_3 + 4y^2A_2 + 6yA_1 + 4A_0)$$
$$Y_{28} = -y^3(y^2A_2 + 5yA_1 + 10A_0)$$
$$Y_{29} = y^5(yA_1 + 6A_0)$$
$$Y_{2(10)} = -y^7A_0$$

ROW 3

$$Y_{32} = -1$$
$$Y_{33} = y^2 - yA_9 + A_8$$
$$Y_{34} = -A_6$$
$$Y_{35} = yA_5 + A_4$$
$$Y_{36} = -(y^2A_4 + 2yA_3 + A_2)$$
$$Y_{37} = y^3A_3 + 3y^2A_2 + 3yA_1 + A_0$$
$$Y_{38} = -y^2(y^2A_2 + 4yA_1 + 6A_0)$$
$$Y_{39} = y^4(yA_1 + 5A_0)$$
$$Y_{3(10)} = -y^6A_0$$

ROW 4

$$Y_{43} = -(2y - A_9)$$
$$Y_{44} = y^3 - y^2A_9 + yA_8 - A_7$$
$$Y_{45} = A_5$$
$$Y_{46} = -(yA_4 + A_3)$$
$$Y_{47} = y^2A_3 + 2yA_2 + A_1$$
$$Y_{48} = -y(y^2A_2 + 3yA_1 + 3A_0)$$
$$Y_{49} = y^3(yA_1 + 4A_0)$$
$$Y_{4(10)} = -y^5A_0$$

ROW 5

$$Y_{53} = +1$$
$$Y_{54} = -(3y^2 - 2yA_9 + A_8)$$
$$Y_{55} = y^4 - y^3A_9 + y^2A_8 - yA_7 + A_6$$
$$Y_{56} = -A_4$$
$$Y_{57} = yA_3 + A_2$$
$$Y_{58} = -(y^2A_2 + 2yA_1 + A_0)$$
$$Y_{59} = y^2(yA_1 + 3A_0)$$
$$Y_{5(10)} = -y^4A_0$$

TABULATION 10-Y

ROW 6

$$Y_{64} = 3y - A_9$$

$$Y_{65} = -(4y^3 - 3y^2A_9 + 2yA_8 - A_7)$$

$$Y_{66} = y^5 - y^4A_9 + y^3A_8 - y^2A_7 + yA_6 - A_5$$

$$Y_{67} = A_3$$

$$Y_{68} = -(yA_2 + A_1)$$

$$Y_{69} = y(yA_1 + 2A_0)$$

$$Y_{6(10)} = -y^3A_0$$

ROW 7

$$Y_{74} = -1$$

$$Y_{75} = 6y^2 - 3yA_9 + A_8$$

$$Y_{76} = -(5y^4 - 4y^3A_9 + 3y^2A_8 - 2yA_7 + A_6)$$

$$Y_{77} = y^6 - y^5A_9 + y^4A_8 - y^3A_7 + y^2A_6 - yA_5 + A_4$$

$$Y_{78} = -A_2$$

$$Y_{79} = yA_1 + A_0$$

$$Y_{7(10)} = -y^2A_0$$

ROW 8

$$Y_{85} = -(4y - A_9)$$

$$Y_{86} = 10y^3 - 6y^2A_9 + 3yA_8 - A_7$$

$$Y_{87} = -(6y^5 - 5y^4A_9 + 4y^3A_8 - 3y^2A_7 + 2yA_6 - A_5)$$

$$Y_{88} = y^7 - y^6A_9 + y^5A_8 - y^4A_7 + y^3A_6 - y^2A_5 + yA_4 - A_3$$

$$Y_{89} = A_1$$

$$Y_{8(10)} = -yA_0$$

ROW 9

$$Y_{95} = +1$$

$$Y_{96} = -(10y^2 - 4yA_9 + A_8)$$

$$Y_{97} = 15y^4 - 10y^3A_9 + 6y^2A_8 - 3yA_7 + A_6$$

$$Y_{98} = -(7y^6 - 6y^5A_9 + 5y^4A_8 - 4y^3A_7 + 3y^2A_6 - 2yA_5 + A_4$$

$$Y_{99} = y^8 - y^7A_9 + y^6A_8 - y^5A_7 + y^4A_6 - y^3A_5 + y^2A_4 - yA_3 + A_2$$

$$Y_{9(10)} = -A_0$$

ROW 10

$$Y_{(10)6} = 5y - A_9$$

$$Y_{(10)7} = -(20y^3 - 10y^2A_9 + 4yA_8 - A_7)$$

$$Y_{(10)8} = 21y^5 - 15y^4A_9 + 10y^3A_8 - 6y^2A_7 + 3yA_6 - A_5$$

$$Y_{(10)9} = -(8y^7 - 7y^6A_9 + 6y^5A_8 - 5y^4A_7 + 4y^3A_6 - 3y^2A_5 + 2yA_4 - A_3)$$

$$Y_{(10)(10)} = y^9 - y^8A_9 + y^7A_8 - y^6A_7 + y^5A_6 - y^4A_5 + y^3A_4 - y^2A_3 + yA_2 - A_1$$

ORDER 12

Polynomial $f(z) = z^{12} + A_{11} z^{11} + A_{10} z^{10} + \cdots + A_1 z + A_0$

Z- Matrix (elements see TABULATION 12-Z)

$$
\begin{bmatrix}
z_{00} & z_{01} & z_{02} & z_{03} & z_{04} & z_{05} & z_{06} & z_{07} & z_{08} & z_{09} & z_{0,10} & z_{0,11} & z_{0,12} \\
z_{10} & z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} & z_{17} & z_{18} & z_{19} & z_{1,10} & z_{1,11} & z_{1,12} \\
z_{20} & z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} & z_{27} & z_{28} & z_{29} & z_{2,10} & z_{2,11} & z_{2,12} \\
z_{30} & z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} & z_{37} & z_{38} & z_{39} & z_{3,10} & z_{3,11} & z_{3,12} \\
z_{40} & z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} & z_{47} & z_{48} & z_{49} & z_{4,10} & z_{4,11} & z_{4,12} \\
z_{50} & z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} & z_{57} & z_{58} & z_{59} & z_{5,10} & z_{5,11} & z_{5,12} \\
z_{60} & z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} & z_{67} & z_{68} & z_{69} & z_{6,10} & z_{6,11} & z_{6,12} \\
z_{70} & z_{71} & z_{72} & z_{73} & z_{74} & z_{75} & z_{76} & z_{77} & z_{78} & z_{79} & z_{7,10} & z_{7,11} & z_{7,12} \\
z_{80} & z_{81} & z_{82} & z_{83} & z_{84} & z_{85} & z_{86} & z_{87} & z_{88} & z_{89} & z_{8,10} & z_{8,11} & z_{8,12} \\
z_{90} & z_{91} & z_{92} & z_{93} & z_{94} & z_{95} & z_{96} & z_{97} & z_{98} & z_{99} & z_{9,10} & z_{9,11} & z_{9,12} \\
z_{10,0} & z_{10,1} & z_{10,2} & z_{10,3} & z_{10,4} & z_{10,5} & z_{10,6} & z_{10,7} & z_{10,8} & z_{10,9} & z_{10,10} & z_{10,11} & z_{10,12} \\
z_{11,0} & z_{11,1} & z_{11,2} & z_{11,3} & z_{11,4} & z_{11,5} & z_{11,6} & z_{11,7} & z_{11,8} & z_{11,9} & z_{11,10} & z_{11,11} & z_{11,12} \\
z_{12,0} & z_{12,1} & z_{12,2} & z_{12,3} & z_{12,4} & z_{12,5} & z_{12,6} & z_{12,7} & z_{12,8} & z_{12,9} & z_{12,10} & z_{12,11} & z_{12,12}
\end{bmatrix}
$$

Y-Matrix (elements see TABULATION 12-Y)

$$
\begin{bmatrix}
Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} & Y_{05} & Y_{06} & Y_{07} & Y_{08} & Y_{09} & Y_{0,10} & Y_{0,11} & Y_{0,12} \\
0 & Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & Y_{17} & Y_{18} & Y_{19} & Y_{1,10} & Y_{1,11} & Y_{1,12} \\
0 & 0 & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26} & Y_{27} & Y_{23} & Y_{29} & Y_{2,10} & Y_{2,11} & Y_{2,12} \\
0 & 0 & Y_{32} & Y_{33} & Y_{34} & Y_{35} & Y_{36} & Y_{37} & Y_{38} & Y_{39} & Y_{3,10} & Y_{3,11} & Y_{3,12} \\
0 & 0 & 0 & Y_{43} & Y_{44} & Y_{45} & Y_{46} & Y_{47} & Y_{48} & Y_{49} & Y_{4,10} & Y_{4,11} & Y_{4,12} \\
0 & 0 & 0 & Y_{53} & Y_{54} & Y_{55} & Y_{56} & Y_{57} & Y_{58} & Y_{59} & Y_{5,10} & Y_{5,11} & Y_{5,12} \\
0 & 0 & 0 & 0 & Y_{64} & Y_{65} & Y_{66} & Y_{67} & Y_{68} & Y_{69} & Y_{6,10} & Y_{6,11} & Y_{6,12} \\
0 & 0 & 0 & 0 & Y_{74} & Y_{75} & Y_{76} & Y_{77} & Y_{78} & Y_{79} & Y_{7,10} & Y_{7,11} & Y_{7,12} \\
0 & 0 & 0 & 0 & 0 & Y_{85} & Y_{86} & Y_{87} & Y_{88} & Y_{89} & Y_{8,10} & Y_{8,11} & Y_{8,12} \\
0 & 0 & 0 & 0 & 0 & Y_{95} & Y_{96} & Y_{97} & Y_{98} & Y_{99} & Y_{9,10} & Y_{9,11} & Y_{9,12} \\
0 & 0 & 0 & 0 & 0 & 0 & Y_{10,6} & Y_{10,7} & Y_{10,8} & Y_{10,9} & Y_{10,10} & Y_{10,11} & Y_{10,12} \\
0 & 0 & 0 & 0 & 0 & 0 & Y_{11,6} & Y_{11,7} & Y_{11,8} & Y_{11,9} & Y_{11,10} & Y_{11,11} & Y_{11,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_{12,7} & Y_{12,8} & Y_{12,9} & Y_{12,10} & Y_{12,11} & Y_{12,12}
\end{bmatrix}
$$

TABULATION 12-Z

ROW 0

$z_{01} = z_{10} = -x^{11}$

$z_{02} = - z_{20} = x^{10}y$

$z_{03} = z_{30} = -x^9(y^2 - x)$

$z_{04} = - z_{40} = x^8y(y^2 - 2x)$

$z_{05} = z_{50} = -x^7(x^2 - 3xy^2 + y^4)$

$z_{06} = - z_{60} = x^6y(3x^2 - 4xy^2 + y^4)$

$z_{07} = z_{70} = -x^5(y^6 - 5xy^4 + 6x^2y^2 - x^3)$

$z_{08} = - z_{80} = x^4y(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$

$z_{09} = z_{90} = -x^3(y^8 - 7xy^6 + 15x^2y^4 - 10x^3y^2 + x^4)$

$z_{0\cdot 10} = -z_{10\cdot 0} = x^2y(y^8 - 8xy^6 + 21x^2y^4 - 20x^3y^2 + 5x^4)$

$z_{0\cdot 11} = z_{11\cdot 0} = -x(y^{10} - 9xy^8 + 28x^2y^6 - 35x^3y^4 + 15x^4y^2 - x^5)$

$z_{0\cdot 12} = -z_{12\cdot 0} = y(y^{10} - 10xy^8 + 36x^2y^6 - 56x^3y^4 + 35x^4y^2 - 6x^5)$

ROW 1

$z_{12} = z_{21} = x^{10}$

$z_{13} = - z_{31} = - x^9y$

$z_{14} = z_{41} = x^8(y^2 - x)$

$z_{15} = - z_{51} = -x^7y(y^2 - 2x)$

$z_{16} = z_{61} = x^6(x^2 - 3xy^2 + y^4)$

$z_{17} = - z_{71} = -x^5y(3x^2 - 4xy^2 + y^4)$

$z_{18} = z_{81} = x^4(y^6 - 5xy^4 + 6x^2y^2 - x^3)$

$z_{19} = - z_{91} = -x^3y(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$

$z_{1\cdot 10} = z_{10\cdot 1} = x^2(y^8 - 7xy^6 + 15x^2y^4 - 10x^3y^2 + x^4)$

$z_{1\cdot 11} = -z_{11\cdot 1} = -xy(y^8 - 8xy^6 + 21x^2y^4 - 20x^3y^2 + 5y^4)$

$z_{1\cdot 12} = z_{12\cdot 1} = y^{10} - 9xy^8 + 28x^2y^6 - 35x^3y^4 + 15x^4y^2 - x^5$

ROW 2

$z_{23} = z_{32} = - x^9$

$z_{24} = - z_{42} = x^8y$

$z_{25} = z_{52} = -x^7(y^2 - x)$

$z_{26} = - z_{62} = x^6y(y^2 - 2x)$

$z_{27} = z_{72} = -x^5(x^2 - 3xy^2 + y^4)$

$z_{28} = - z_{82} = x^4y(3x^2 - 4xy^2 + y^4)$

$z_{29} = z_{92} = -x^3(y^6 - 5xy^4 + 6x^2y^2 - x^3)$

$z_{2\cdot 10} = -z_{10\cdot 2} = x^2y(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$

$z_{2\cdot 11} = z_{11\cdot 2} = -x(y^8 - 7xy^6 + 15x^2y^4 - 10x^3y^2 + x^4)$

$z_{2\cdot 12} = -z_{12\cdot 2} = y(y^8 - 8xy^6 + 21x^2y^4 - 20x^3y^2 + 5y^4)$

ROW 3

$z_{34} = z_{43} = x^8$

$z_{35} = - z_{53} = - x^7y$

$z_{36} = z_{63} = x^6(y^2 - x)$

$z_{37} = - z_{73} = -x^5y(y^2 - 2x)$

$z_{38} = z_{83} = x^4(x^2 - 3xy^2 + y^4)$

$z_{39} = - z_{93} = -x^3y(3x^2 - 4xy^2 + y^4)$

$z_{3\cdot 10} = z_{10\cdot 3} = x^2(y^6 - 5xy^4 + 6x^2y^2 - x^3)$

$z_{3\cdot 11} = - z_{11\cdot 3} = -xy(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$

$z_{3\cdot 12} = z_{12\cdot 3} = y^8 - 7xy^6 + 15x^2y^4 - 10x^3y^2 + x^4$

ROW 4

$z_{45} = z_{54} = - x^7$

$z_{46} = - z_{64} = x^6y$

$z_{47} = z_{74} = -x^5(y^2 - x)$

$z_{48} = - z_{84} = x^4y(y^2 - 2x)$

$z_{49} = z_{94} = -x^3(x^2 - 3xy^2 + y^4)$

$z_{4\cdot 10} = -z_{10\cdot 4} = x^2y(3x^2 - 4xy^2 + y^4)$

$z_{4\cdot 11} = z_{11\cdot 4} = -x(y^6 - 5xy^4 + 6x^2y^2 - x^3)$

$z_{4\cdot 12} = -z_{12\cdot 4} = y(y^6 - 6xy^4 + 10x^2y^2 - 4x^3)$

ROW 5

$z_{56} = z_{65} = x^6$

$z_{57} = - z_{75} = -x^5y$

$z_{58} = z_{85} = x^4(y^2 - x)$

$z_{59} = - z_{95} = -x^3y(y^2 - 2x)$

$z_{5\cdot 10} = z_{10\cdot 5} = x^2(x^2 - 3xy^2 + y^4)$

$z_{5\cdot 11} = -z_{11\cdot 5} = -xy(3x^2 - 4xy^2 + y^4)$

$z_{5\cdot 12} = z_{12\cdot 5} = y^6 - 5xy^4 + 6x^2y^2 - x^3$

TABULATION 12-Z

ROW 6

$$z_{67} = z_{76} = -x^5$$

$$z_{68} = -z_{86} = x^4y$$

$$z_{69} = z_{96} = -x^3(y^2 - x)$$

$$z_{6,10} = -z_{10,6} = x^2y(y^2 - 2x)$$

$$z_{6,11} = z_{11,6} = -x(x^2 - 3xy^2 + y^4)$$

$$z_{6,12} = -z_{12,6} = y(3x^2 - 4xy^2 + y^4)$$

ROW 7

$$z_{78} = z_{87} = x^4$$

$$z_{79} = -z_{97} = -x^3y$$

$$z_{7,10} = z_{10,7} = x^2(y^2 - x)$$

$$z_{7,11} = -z_{11,7} = -xy(y^2 - 2x)$$

$$z_{7,12} = z_{12,7} = x^2 - 3xy^2 + y^4$$

ROW 8

$$z_{89} = z_{98} = -x^3$$

$$z_{8,10} = -z_{10,8} = x^2y$$

$$z_{8,11} = z_{11,8} = -x(y^2 - x)$$

$$z_{8,12} = -z_{12,8} = y(y^2 - 2x)$$

ROW 9

$$z_{9,10} = z_{10,9} = x^2$$

$$z_{9,11} = -z_{11,9} = -xy$$

$$z_{9,12} = z_{12,9} = y^2 - x$$

ROW 10

$$z_{10,11} = z_{11,10} = -x$$

$$z_{10,12} = -z_{12,10} = y$$

ROW 11

$$z_{11,12} = z_{12,11} = +1$$

TABULATION 12-Y

ROW 0

$Y_{00} = 0$

$Y_{01} = A_{11}$

$Y_{02} = -(yA_{10} + A_9)$

$Y_{03} = y^2A_9 + 2yA_8 + A_7$

$Y_{04} = -(y^3A_8 + 3y^2 A_7 + 3yA_6 + A_5)$

$Y_{05} = y^4A_7 + 4y^3A_6 + 6y^2A_5 + 4yA_4 + A_3$

$Y_{06} = -(y^5A_6 + 5y^4A_5 + 10y^3A_4 + 10y^2A_3 + 5yA_2 + A_1)$

$Y_{07} = y(y^5A_5 + 6y^4A_4 + 15y^3A_3 + 20y^2A_2 + 15yA_1 + 6A_0)$

$Y_{08} = -y^3(y^4A_4 + 7y^3A_3 + 21y^2A_2 + 35yA_1 + 35A_0)$

$Y_{09} = y^5(y^3A_3 + 8y^2A_2 + 28yA_1 + 56A_0)$

$Y_{0,10} = -y^7(y^2A_2 + 9yA_1 + 36A_0)$

$Y_{0,11} = y^9(yA_1 + 10A_0)$

$Y_{0,12} = -y^{11}A_0$

ROW 1

$Y_{11} = +1$

$Y_{12} = -A_{10}$

$Y_{13} = yA_9 + A_8$

$Y_{14} = -(y^2A_8 + 2yA_7 + A_6)$

$Y_{15} = y^3A_7 + 3y^2A_6 + 3yA_5 + A_4$

$Y_{16} = -(y^4A_6 + 4y^3A_5 + 6y^2A_4 + 4yA_3 + A_2)$

$Y_{17} = y^5A_5 + 5y^4A_4 + 10y^3A_3 + 10y^2A_2 + 5yA_1 + A_0$

$Y_{18} = -y^2(y^4A_4 + 6y^3A_3 + 15y^2A_2 + 20yA_1 + 15A_0)$

$Y_{19} = y^4(y^3A_3 + 7y^2A_2 + 21yA_1 + 35A_0)$

$Y_{1,10} = -y^6(y^2A_2 + 8yA_1 + 28A_0)$

$Y_{1,11} = y^8(yA_1 + 9A_0)$

$Y_{1,12} = -y^{10}A_0$

ROW 2

$Y_{22} = y - A_{11}$

$Y_{23} = A_9$

$Y_{24} = -(yA_8 + A_7)$

$Y_{25} = y^2A_7 + 2yA_6 + A_5$

$Y_{26} = -(y^3A_6 + 3y^2A_5 + 3yA_4 + A_3)$

$Y_{27} = y^4A_5 + 4y^3A_4 + 6y^2A_3 + 4yA_2 + A_1$

$Y_{28} = -y(y^4A_4 + 5y^3A_3 + 10y^2A_2 + 10yA_1 + 5A_0)$

$Y_{29} = y^3(y^3A_3 + 6y^2A_2 + 15yA_1 + 20A_0)$

$Y_{2,10} = -y^5(y^2A_2 + 7yA_1 + 21A_0)$

$Y_{2,11} = y^7(yA_7 + 8A_0)$

$Y_{2,12} = -y^9A_0$

ROW 3

$Y_{32} = -1$

$Y_{33} = y^2 - yA_{11} + A_{10}$

$Y_{34} = -A_8$

$Y_{35} = yA_7 + A_6$

$Y_{36} = -(y^2A_6 + 2yA_5 + A_4)$

$Y_{37} = y^3A_5 + 3y^2A_4 + 3yA_3 + A_2$

$Y_{38} = -(y^4A_4 + 4y^3A_3 + 6y^2A_2 + 4yA_1 + A_0)$

$Y_{39} = y^2(y^3A_3 + 5y^2A_2 + 10yA_1 + 10A_0)$

$Y_{3,10} = -y^4(y^2A_2 + 6yA_1 + 15A_0)$

$Y_{3,11} = y^6(yA_1 + 7A_0)$

$Y_{3,12} = -y^8A_0$

ROW 4

$Y_{43} = -(2y - A_{11})$

$Y_{44} = y^3 - y^2A_{11} + yA_{10} - A_9$

$Y_{45} = A_7$

$Y_{46} = -(yA_6 + A_5)$

$Y_{47} = y^2A_5 + 2yA_4 + A_3$

$Y_{48} = -(y^3A_4 + 3y^2A_3 + 3yA_2 + A$[illegible]$)$

$Y_{49} = y(y^3A_3 + 4y^2A_2 + 6yA_1 + 4A_0)$

$Y_{4,10} = -y^3(y^2A_2 + 5yA_1 + 10A_0)$

$Y_{4,11} = y^5(yA_1 + 6A_0)$

$Y_{4,12} = -y^7A_0$

ROW 5

$Y_{53} = +1$

$Y_{54} = -(3y^2 - 2yA_{11} + A_{10})$

$Y_{55} = y^4 - y^3A_{11} + y^2A_{10} - yA_9 + A_8$

$Y_{56} = -A_6$

$Y_{57} = yA_5 + A_4$

$Y_{58} = -(y^2A_4 + 2yA_3 + A_2)$

$Y_{59} = y^3A_3 + 3y^2A_2 + 3yA_1 + A_0$

$Y_{5,10} = -y^2(y^2A_2 + 4yA_1 + 6A_0)$

$Y_{5,11} = y^4(yA_1 + 5A_0)$

$Y_{5,12} = -y^6A_0$

TABULATION 12-Y

ROW 6

$$Y_{64} = 3y - A_{11}$$

$$Y_{65} = -(4y^3 - 3y^2A_{11} + 2yA_{10} - A_9)$$

$$Y_{66} = y^5 - y^4A_{11} + y^3A_{10} - y^2A_9 + yA_8 - A_7$$

$$Y_{67} = A_5$$

$$Y_{68} = -(yA_4 + A_3)$$

$$Y_{69} = y^2A_3 + 2yA_2 + A_1$$

$$Y_{6,10} = -y(y^2A_2 + 3yA_1 + 3A_0)$$

$$Y_{6,11} = y^3(yA_1 + 4A_0)$$

$$Y_{6,12} = -\,y^5A_0$$

ROW 7

$$Y_{74} = -\,1$$

$$Y_{75} = 6y^2 - 3yA_{11} + A_{10}$$

$$Y_{76} = -(5y^4 - 4y^3A_{11} + 3y^2A_{10} - 2yA_9 + A_8)$$

$$Y_{77} = y^6 - y^5A_{11} + y^4A_{10} - y^3A_9 + y^2A_8 - yA_7 + A_6$$

$$Y_{78} = -\,A_4$$

$$Y_{79} = yA_3 + A_2$$

$$Y_{7,10} = -(y^2A_2 + 2yA_1 + A_0)$$

$$Y_{7,11} = y^2(yA_1 + 3A_0)$$

$$Y_{7,12} = -\,y^4A_0$$

ROW 8

$$Y_{85} = -\,(4y - A_{11})$$

$$Y_{86} = 10y^3 - 6y^2A_{11} + 3yA_{10} - A_9$$

$$Y_{87} = -(6y^5 - 5y^4A_{11} + 4y^3A_{10} - 3y^2A_9 + 2yA_8 - A_7)$$

$$Y_{88} = y^7 - y^6A_{11} + y^5A_{10} - y^4A_9 + y^3A_8 - y^2A_7 + yA_6 - A_5$$

$$Y_{89} = A_3$$

$$Y_{8,10} = -(yA_2 + A_1)$$

$$Y_{8,11} = y(yA_1 + 2A_0)$$

$$Y_{8,12} = -\,y^3A_0$$

TABULATION 12-Y

ROW 9

$Y_{95} = +1$

$Y_{96} = -(10y^2 - 4yA_{11} + A_{10})$

$Y_{97} = 15y^4 - 10y^3A_{11} + 6y^2A_{10} - 3yA_9 + A_8$

$Y_{98} = -(7y^6 - 6y^5A_{11} + 5y^4A_{10} - 4y^3A_9 + 3y^2A_8 - 2yA_7 + A_6)$

$Y_{99} = y^8 - y^7A_{11} + y^6A_{10} - y^5A_9 + y^4A_8 - y^3A_7 + y^2A_6 - yA_5 + A_4$

$Y_{9\cdot 10} = -A_2$

$Y_{9\cdot 11} = yA_1 + A_0$

$Y_{9\cdot 12} = -y^2A_0$

ROW 10

$Y_{10\cdot 6} = 5y - A_{11}$

$Y_{10\cdot 7} = -(20y^3 - 10y^2A_{11} + 4yA_{10} - A_9)$

$Y_{10\cdot 8} = 21y^5 - 15y^4A_{11} + 10y^3A_{10} - 6y^2A_9 + 3yA_8 - A_7$

$Y_{10\cdot 9} = -(8y^7 - 7y^6A_{11} + 6y^5A_{10} - 5y^4A_9 + 4y^3A_8 - 3y^2A_7 + 2yA_6 - A_5)$

$Y_{10\cdot 10} = y^9 - y^8A_{11} + y^7A_{10} - y^6A_9 + y^5A_8 - y^4A_7 + y^3A_6 - y^2A_5 + yA_4 - A_3$

$Y_{10\cdot 11} = A_1$

$Y_{10\cdot 12} = -yA_0$

ROW 11

$Y_{11\cdot 6} = -1$

$Y_{11\cdot 7} = 15y^2 - 5yA_{11} + A_{10}$

$Y_{11\cdot 8} = -(35y^4 - 20y^3A_{11} + 10y^2A_{10} - 4yA_9 + A_8)$

$Y_{11\cdot 9} = 28y^6 - 21y^5A_{11} + 15y^4A_{10} - 10y^3A_9 + 6y^2A_8 - 3yA_7 + A_6$

$Y_{11\cdot 10} = -(9y^8 - 8y^7A_{11} + 7y^6A_{10} - 6y^5A_9 + 5y^4A_8 - 4y^3A_7 + 3y^2A_6 - 2yA_5 + A_4)$

$Y_{11\cdot 11} = y^{10} - y^9A_{11} + y^8A_{10} - y^7A_9 + y^6A_8 - y^5A_7 + y^4A_6 - y^3A_5 + y^2A_4 - yA_3 + A_2$

$Y_{11\cdot 12} = -A_0$

ROW 12

$Y_{12\cdot 7} = -(6y - A_{11})$

$Y_{12\cdot 8} = 35y^3 - 15y^2A_{11} + 5yA_{10} - A_9$

$Y_{12\cdot 9} = -(56y^5 - 35y^4A_{11} + 20y^3A_{10} - 10y^2A_9 + 4yA_8 - A_7)$

$Y_{12\cdot 10} = 36y^7 - 28y^6A_{11} + 21y^5A_{10} - 15y^4A_9 + 10y^3A_8 - 6y^2A_7 + 3yA_6 - A_5$

$Y_{12\cdot 11} = -(10y^9 - 9y^8A_{11} + 8y^7A_{10} - 7y^6A_9 + 6y^5A_8 - 5y^4A_7 + 4y^3A_6 - 3y^2A_5 + 2yA_4 - A_3)$

$Y_{12\cdot 12} = y^{11} - y^{10}A_{11} + y^9A_{10} - y^8A_9 + y^7A_8 - y^6A_7 + y^5A_6 - y^4A_5 + y^3A_4 - y^2A_3 + yA_2 - A_1$

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ON TWO MATRIX SYSTEMS DERIVED FROM A POLYNOMIAL OF EVEN ORDER WITH REAL COEFFICIENTS, by Kurt H. Haase. December 1961 51 pp incl. tables. AFCRL 983
Unclassified report

A polynomial f(z) of even order n and real coefficients $A_n = 1, A_{n-1}, \ldots, A_1, A_0$, has n/2 pairs of roots, each pair yielding a square polynomial $z^2 + y_i z + x_i$ with real coefficients y_i and x_i. Hence there is a summation form and a product form for f(z). From a comparison of the coefficients for any power of z, n+1 equations can be derived, each consistent for any pair y_i and x_i. These equations involve the elements Z_{ik}^{1} of an $(n+1) \times (n+1)$ square matrix with columns multiplied by the A coefficients. Simultaneous polynomials in x and y defined as real variables, the elements of this 'Z matrix' depend only on the order n of f(z), (over)

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not on the coefficients, and are known for any even order. A transcription of the derived equations yields a Y matrix whose elements are multiplied by powers of x, or an X matrix whose elements are multiplied by powers of y. The elements of the Y matrix are polynomials in y only, with real coefficients that are combinations of the A coefficients. The Y matrix is immediately known for any order n of f(z). The properties of these derived matrices are compiled in 12 theorems.

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